Isodiametric problem in the spherical and hyperbolic spaces

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Introduction

Let $\mathcal{M}^n$ be either the Euclidean space $\mathbb{R}^n$, hyperbolic space H^n or spherical space S^n for $n \geq 2$. We write $V_{\mathcal{M}^n}$ to denote the n-dimensional Lebesgue measure on $\mathcal{M}^n$, and $d_{\mathcal{M}^n}(x,y)$ to denote the geodesic distance between $x, y \in \mathcal{M}^n$.
For a bounded set $X \subset \mathcal{M}^n$, its diameter $\mathrm{diam}_{\mathcal{M}^n} X$ is the supremum of the geodesic distances $d_{\mathcal{M}^n}(x,y)$ for $x, y \in X$. For $D > 0$ and $n \geq 2$, our goal is to determine the maximal volume of a subset of $\mathcal{M}^n$ of diameter at most D. For any $z \in \mathcal{M}^n$ and $r > 0$, let

$$B_{\mathcal{M}^n}(z,r) = \{x \in \mathcal{M}^n : d_{\mathcal{M}^n}(x,z) \leq r\}$$

be the n-dimensional ball centered at z where it is natural to assume $r < \pi$ if $\mathcal{M}^n = S^n$.
The isodiametric problem was solved in $\mathbb{R}^2$ by *Bieberbach*[2], while *Urysohn*[4] solved the n-dimensional Euclidean case. The following generalization is by *Böröczky* and *Sagmeister*:

Isodiametric inequality [3]

If $\mathcal{M}^n$ is either $\mathbb{R}^n$, S^n or H^n, $D > 0$ (with $D < \pi$ if $\mathcal{M}^n = S^n$) and $X \subset \mathcal{M}^n$ is measurable and bounded with $\mathrm{diam}_{\mathcal{M}^n} X \leq D$, then

$$V_{\mathcal{M}^n}(X) \leq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}(z_0, D/2)\right),$$

and equality holds if and only if the closure of X is a ball of radius $D/2$.

Isodiametric stability

If $\mathcal{M}^n$ is either $\mathbb{R}^n$, S^n or H^n, $D > 0$ (where $D < \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$) and $X \subset \mathcal{M}^n$ is measurable with $\mathrm{diam}_{\mathcal{M}^n} X \leq D$ and

$$V_{\mathcal{M}^n}(X) \geq V_{\mathcal{M}^n}\left(B_{\mathcal{M}^n}(z_0, D/2)\right) - \varepsilon,$$

then there exists a $y_0 \in \mathcal{M}^n$ such that

$$V_{\mathcal{M}^n}\left(\mathrm{conv}_{\mathcal{M}^n}(X) \triangle B_{\mathcal{M}^n}\left(y_0, \tfrac{D}{2}\right)\right) \leq \gamma_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}}$$

and

$$B_{\mathcal{M}^n}\left(z, \tfrac{D}{2} - \tilde{\gamma}_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}}\right) \subseteq \mathrm{conv}_{\mathcal{M}^n}(X)$$

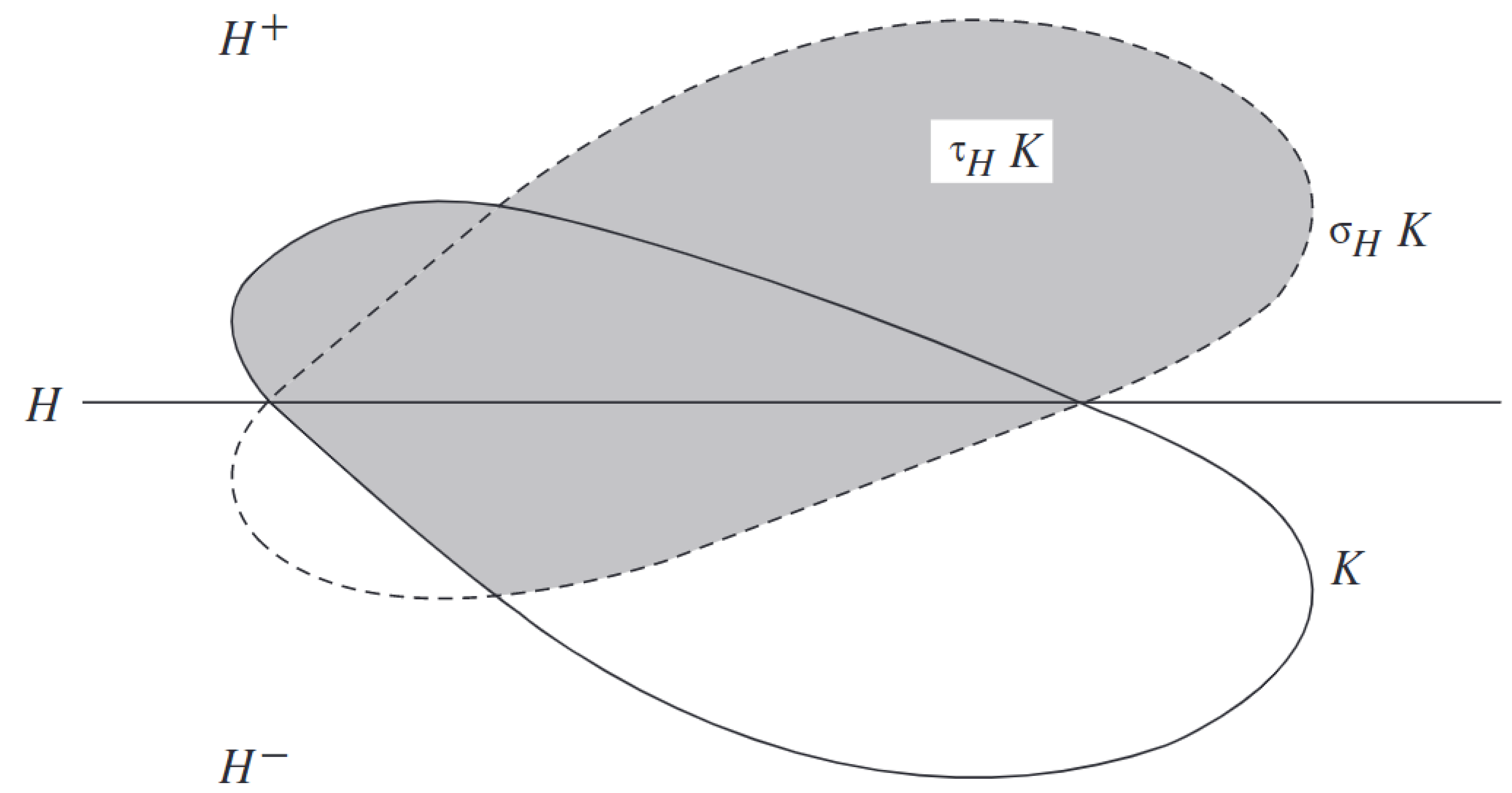
and also

$$\mathrm{conv}_{\mathcal{M}^n}(X) \subseteq B_{\mathcal{M}^n}\left(z, \tfrac{D}{2} + \tilde{\gamma}_{\mathcal{M}^n}(D) \cdot \varepsilon^{\frac{2}{3n+2}}\right),$$

where $\mathrm{conv}_{\mathcal{M}^n}(X)$ denotes the convex hull of X in $\mathcal{M}^n$.
Partially similar results have already been proved for the non-convex case.

Two-point symmetrization

Let H^+ be a closed halfspace bounded by the $(n-1)$-dimensional subspace H in $\mathcal{M}^n$, and let $X \subset \mathcal{M}^n$ be compact. We write H^- to denote the other closed halfspace of $\mathcal{M}^n$ determined by H and $\sigma_H X$ to denote the reflected image of X through the $(n-1)$-subspace H. The two-point symmetrization $\tau_{H^+} X$ of X with respect to H^+ is a rearrangement of X by replacing $(H^- \cap X) \setminus \sigma_H X$ by its reflected image through H where readily this reflected image is disjoint from X.



Two-point symmetrization can be effectively used for the isodiametric problem, as it preserves Lebesgue measure while not increasing diameter. *Aubrun* and *Fradelizi* showed[1] for $\mathcal{M}^n = \mathbb{R}^n$ and $\mathcal{M}^n = H^n$, that $\tau_{H^+} X$ is convex for a convex body X for any H hyperplane iff X is a ball. This is also true in the spherical case (see[3]). Two-point symmetrization also played a key role proving the stability versions. We investigated parallel domains of convex bodies as they have a nice boundary.

References

[1] G. Aubrun, M. Fradelizi, *Two-point symmetrization and convexity*, Arch. Math., 82 (2004), 282-288.

[2] L. Bierbach *Über eine Extremaleigenschaft des Kreises*, Jber. Deutsch. Math.-Verein., 24 (1915), 247-250.

[3] K. J. Böröczky, Á. Sagmeister,*The isodiametric problem on the sphere and in the hyperbolic space*, Acta Math. Hungar. 160, 13–32 (2020).

[4] P. Urysohn, *Mittlere Breite und Volumen der konvexen Körper im n-dimensionalen Raume*, Matem. Sb. SSSR 31 (1924), 477-486.t