GENERALIZATIONS OF KLAIN'S THEOREM TO COMPACT SETS

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OBJECTIVES

We want to extend the knwoledge and application range of certain kind of symmetrizations, namely extending the validity of some result in the convex case to the compact case. Given a class $\mathcal{E}^n$ of subsets of $\mathbb{R}^n$, we define a symmetrization with respect to a subspace H as a map

$$\diamondsuit_H : \mathcal{E}^n \to \mathcal{E}^n_H,$$

where $\mathcal{E}^n_H$ are the H-symmetric elements of $\mathcal{E}^n$.
Main properties, some of which are lost switching to compact sets:

1. $H^\perp$ translation invariance,
2. Monotonicity for some valuation function,
3. Idempotence,
4. Invariance for H symmetry.

COMPACT CHASING CONVEX

Using Shapley-Folkman-Starr Theorem, we obtain the following result.
Let $K \in \mathcal{K}^n$, $(\mathbb{A}_m)_{m\in\mathbb{N}}$ a sequence of isometries. If the sequence of convex sets

$$K_m := \frac{\mathbb{A}_m K + \cdots + \mathbb{A}_1 K}{m}$$

converges to $L \in \mathcal{K}^n$, then every compact set C such that $\mathrm{conv}(C) = K$ converges to L through the same process.
This results implies immediately Klain's Theorem for Minkowski symmetrization of compact sets.
Moreover, this tells us that Hadwiger's Theorem on the convergence to the ball is valid for compact sets too. This last result in particular, allows us to extend the validity of and extimate result from Salani for viscosity solutions of elliptic PDEs.

A KLAIN-LIKE RESULT

Let K in $\mathcal{E}^n$, $\mathcal{F} = \{Q_1, \dots, Q_k\}$ a **finite family** of subsets of $\mathbb{R}^n$. If $(H_m)_{m\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}^n$ such that for every m $H_m = Q_j$ for some $1 \le j \le k$, then the sequence

$$K_m := \diamondsuit_{H_m} \dots \diamondsuit_{H_1} K$$

converges to a set $L \in \mathcal{E}^n$. Moreover, this set is symmetric with respect to every element of $\mathcal{F}$.

- This was first proved by Klain [1] for Steiner Symmetrization of convex sets.
- Bianchi, Burchard, Gronchi, Volcic [2] proved it for compact sets.
- Bianchi, Gardner, Gronchi [3] generalized this result for general symmetrizations in the convex setting.

IDEMPOTENCE OF SET ADDITION

Consider the one dimensional set

$$L = [a, a+\varepsilon] \cup [b-\varepsilon, b].$$

We prove that for sets of this kind, there exist an idempotece index k depending on the total width of the set and ε such that

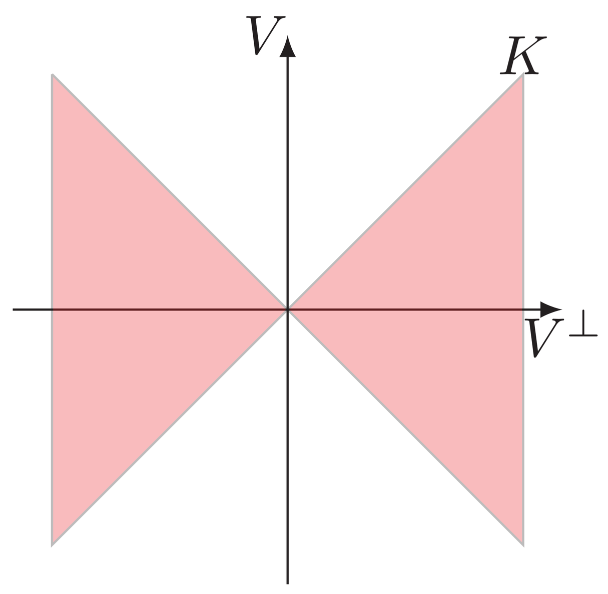
$$\frac{L + \cdots + L}{k} = [a, b].$$

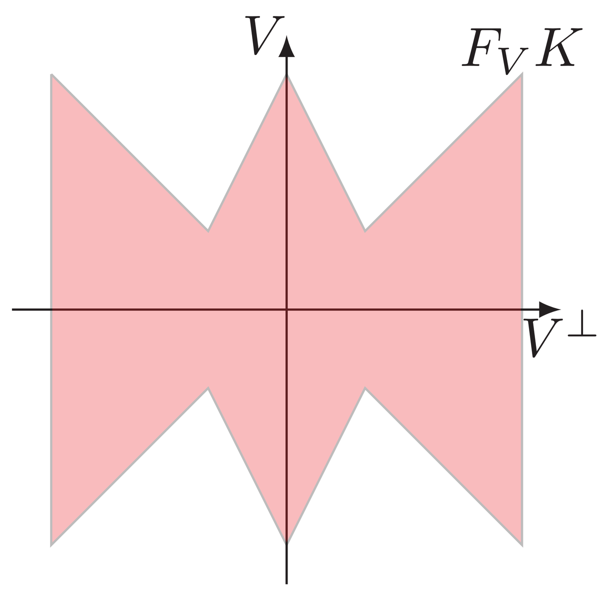
In $\mathbb{R}^n, n \ge 2$, we prove for two compact sets K, L with connected boundary and some intersection condition that

$$K + L = \partial K + \partial L.$$

In particular this will imply that if we have a set K and $K \supseteq \partial\mathrm{conv}(K)$ then $M_H K = M_H \mathrm{conv}(K)$.

OUR SYMMETRIZATIONS AND THE COMPACT CASE





These two symmetrizations behave really well in the convex case, mainly because of the role of Minkowski addition. This is no longer true when we deal with general compact sets. First of all, we lose the invariance for H symmetric sets. A simple example a couple of points symmetric with respect to the vertical axis in $\mathbb{R}^2$. Iterating Minkowski symmetrization, we obtain a sequence of symmetric sets converging to the convex envelope of the two initial points. In the figures a more complicated example is shown.

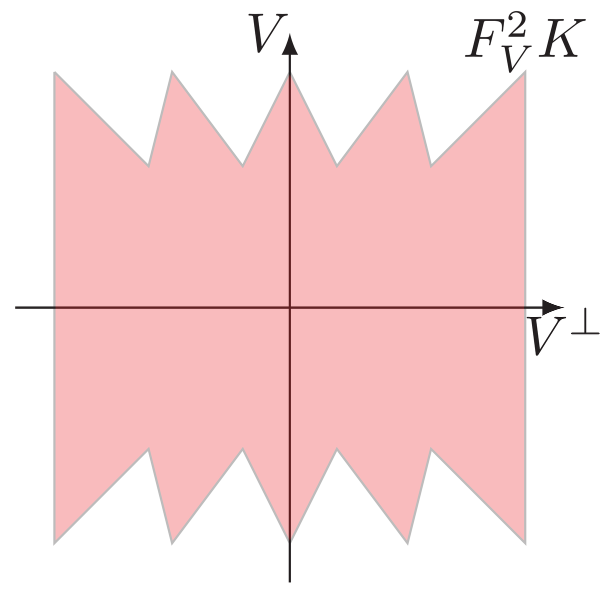
We mainly work with:

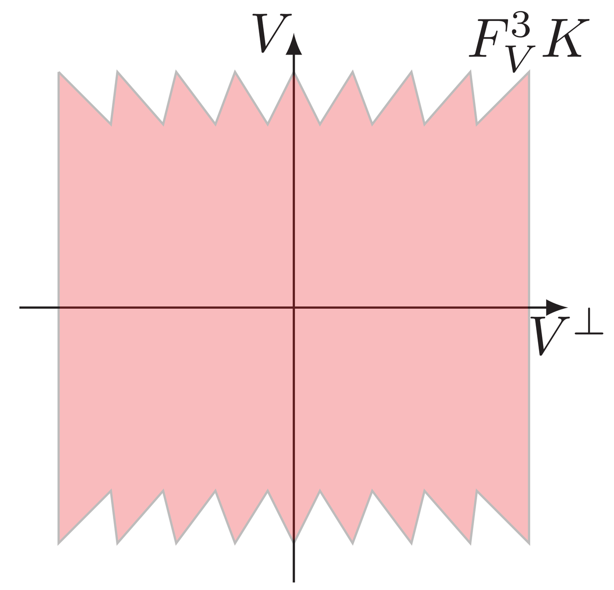
- *Minkowski Symmetrization*:

$$M_H K = \frac{K + R_H K}{2}$$

- *Fiber Symmetrization*:

$$F_H K = \bigcup_{x\in H} M_H(K \cap (H + x))$$





RESULTS FOR CONVEX SHELLS

The preciding results allow us to prove a Klain-like Theorem for Fiber symmetrization of compact sets K such that $K \supseteq \mathrm{conv}(K)$. In particular, if $K \in \mathbb{R}^n, n \ge 2$ and the subspaces of the family $\mathcal{F}$ have dimension $1 \le d \le n-2$, we obtain the same convergence result. In particular, as for the case of Minkowsky symmetrization, we obtain a limit set which is convex.
The dimensional constraints occur because of the behaviour of one dimensional sections. In particular the connectedness of the boundary is crucial for proving the sufficient conditions.
Another application is the extension of a result from Klartag [4].
Let $n \ge 2, 0 < \epsilon < 1/2$, and let $K \subset \mathbb{R}^n$ be a compact set such that $K \supseteq \partial\mathrm{conv}K$. Then there exist $cn \log 1/\epsilon$ Minkowski symmetrizations with respect to hyperplanes, that transform K into a body $\tilde{K}$ that satisfies

$$(1-\epsilon)w(K)B^n \subset \tilde{K} \subset (1+\epsilon)w(K)B^n,$$

where $c > 0$ is some numerical constant.
This results underline the deep connection of the Minkowski addition with the extremals points of the objects we are working with.

FUTURE RESEARCH

- Results for general symmetrizations of compact sets
- Applications to PDEs
- Classification of nonconverging sequences

REFERENCES

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[2] G. Bianchi, A. Burchard, P. Gronchi, A. Volcic, *Convergence in Shape of Steiner Symmetrization*
[3] G. Bianchi, R. J. Gardner, P. Gronchi, *Convergence Of Symmerization Processes*
[4] B. Klartag, *Rate of convergence of geometric symmetrizations*

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