UNIVERSIDAD DE **MURCIA**

FACULTY OF **MATHEMATICS**

Functional Analysis

Syllabus

Course code:	1599
Number of ECTS credits:	6
Semester:	1st (September-January)
Prerequisites:	None
Recommended components:	Having an acquaintance with the following courses is strongly recommended: One-Variable Real Functions I & II (1568, 1573), Linear Algebra (1569), Affine and Euclidean Geometry (1574), Topology of Metric Spaces (1575), Mathematical Analysis in Several Variables I, II & III (1578, 1579, 1583), Further Linear Algebra and Geometry (1580), Complex Variable Functions (1588).
Language of instruction:	Spanish (students are allowed to ask questions and write homeworks and exams in English)

Course description

Functional Analysis provides an appropriate framework to study some optimization problems where the unknown variable is a function (for instance, those of Variational Calculus); thus, in some sense, we are interested in solving linear systems with "infinitely many" variables. From the new perspective of Functional Analysis (and this explains his origin), functions are points in a space with a convenient underlying structure, and existence problems can be formulated in terms of completeness; moreover, solutions can be "geometrically" characterized. Generally speaking, these structures are both vectorial and topological, but in this course we shall be mainly concerned with Banach spaces, with a special emphasis on Hilbert spaces.

The course is divided into three parts. The first one is mainly devoted to Hilbert spaces; among other things, the classical Dirichlet variation problem will be studied. The second part deals with the theory of operators. Here we shall see that some well known results on finite-dimensional matrices can be conveniently extended to operators on Hilbert spaces, and learn how to apply them to some differential equations. The third part is devoted to the fundamental principles of Functional Analysis and its applications.

Learning outcomes and competences

After completion of this course you will:

1. learn how to use projections, decompositions and basis of separable Hilbert spaces.
2. understand the equivalence between the projection, Riesz, and Lax-Milgran theorems and the minimization of strongly positive quadratic forms in a Hilbert space.

3. learn to prove the Dirichlet princle using Hilbert spaces techniques.

4. learn the notion of Hilbert basis and be able to solve equations manipulating coordinates in infinite-dimensional spaces; also you will construct Hilbert basis in (among other function spaces) the space of square-integrable functions and the Bergman space, and understand the L2 theory of Fourier series.

5. deal with concrete examples of operators; they will be chosen either because they have a simple and clear behaviour and can be potentially extended to more complex settings, or because they are important in applications.

6. understand the notion of spectrum of an operator, and learn how to apply it to solve equations.

7. be able to prove the spectral theorem for normal compact operators and the Fredholm alternative theorems.

8. know how to apply the spectral theorem to solve some types of integral equations involving the Sturm-Liouville problem and some partial differential equations which are related to it.

9. learn the Hahn-Banach extension theorem and its usefulness as a tool to deal with abstract problems of approximation and separation of convex sets.

10. understand the Baire category theorem and its consequences in Banach spaces: the uniform boundedness theorem and the closed graph theorem.

Course contents

I. Spaces

   1. Survey of normed vector spaces.

      First examples. Properties of normed spaces. Banach spaces and their completion. Continuity and extensions of linear mappings.

   2. Hilbert spaces.

      Pre-Hilbert spaces. Examples. The parallelogram identity. Further examples: the Bergman space and the space of square-integrable functions.

   3. Optimization in Hilbert spaces.

      Best approximation. The Projection and Riesz theorems. The Lax-Milgram theorem. Application to the Dirichlet principle. Distributional derivative and Sobolev spaces.

   4. Basis in Hilbert spaces.

      Characterization. Basis on function spaces: Fourier series, ortogonal polynomials and basis in the Bergman space. Application to trigonometric Fourier series.

II. Operators

   1. Operators on Banach spaces.

      Basic notions on operators. Operator spaces and ideals. Examples of operators in Banach spaces.

   2. Spectral theory.

      Compact operators. The Riesz theory. Compactness in Hilbert spaces. Spectral theory for self-adjoint and normal operators.

3. Applications.
   Fredholm integral equations. Sturm-Liouville problems.

III. Fundamental principles

1. The Baire theorem.
   A proof of the Baire theorem in complete metric spaces and some applications.
2. The uniform boundedness principle.
   The Banach-Steinhaus theorem. An application: existence of continuous functions whose Fourier series does not converge pointwise.
3. The open mapping theorem.
   The closed graph theorem. Applications.
4. The Hanh-Banach theorem.
   Analytic and geometric consequences. Dual spaces and its calculation in concrete spaces.

References

Main texts

1. Cascales B., Mira J. M., Orihuela J. & Raja M., *Análisis Funcional*; E-Lectolibris, 2013.

Supplementary references

1. Brezis H., *Analyse Fonctionelle, Théorie et Applications*; Masson, 1983.
2. Conway J. B., *A Course in Functional Analysis*; Springer, 1985.
3. Fabian M., Habala P., Hájek P., Montesinos Santalucía V., Pelant J. & Zizler V., *Functional Analysis and Infinite-Dimensional Geometry*; Springer, 2001.
4. Kolmogorov A. N. & Fomin S. V, *Elementos de la Teoría de Funciones y del Análisis Funcional*; MIR, 1975.
5. Maurey B., *Analyse Fonctionelle et Téorie Spectrale*; Classroom notes, Université de Paris VII, 2001.
6. Tocino García A. & Maldonado Cordero M., *Problemas resueltos de Análisis Funcional*; Librería Cervantes, 2003.
7. Weinberger H. F., *Curso de Ecuaciones en Derivadas Parciales*; Reverté, 1988.
8. Zeidler E., *Applied Functional Analysis*; Springer, 1995.