Mathematics and Mathematics Education I: magnitudes and measurement

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Contents

CHAPTER 2

Anthropological Theory of Didactic Phenomena

In this unit we give a short account of some basic notions of the so-called *Anthropological Theory of Didactic Phenomena*(=ATD).

1. Mathematical praxeologies

According to the ATD *didactic* means *related to the study of some questions*. Thus, all what is done in a certain institution in order to face problematic questions deserves the adjective "didactic". Tipically, the study takes place in a *community of study*, following a *study program*, and under the guidance of one or several *directors* of the community of study.

Example 1.1.
- *The researcher undertakes a study activity, typically in a research group (*i.e. *the community of study), following a research program (*i.e. *the study program) under the supervision of one or several main researchers (*i.e. *the director(s) of study).*
- *The student equally undertakes a study activity, tipically in a class (people in the class constitute the community of study), following an official curriculum (*i.e. *the study program) under the supervision of a teacher (*i.e. *the director of study).*

Mathematics can be regarded both as an activity (namely, the study of some problematic questions) and the output of this activity (namely, a certain set of results, called *mathematical knowledge*, which can be organized in several ways).

Example 1.2.
- *Activity: to study how to share out a certain amount equitably (*e.g. 57 *stickers among* 3 *children A, B and C)*
- *Possible resulting mathematical knowledge: a possible technique would be to give first a sticker to A, then 1 sticker to B, then 1 sticker to C, start again with A, and go on until I have no stickers left. (Needless to say, this technique has serious problems, but still it is a technique which sometimes works!)*
- *Another possible resulting mathematical knowledge: to subtract* 3 *to* 57 *as many times as possible, say n times. In this case I have to give n stickers to each child. (This technique always works.)*

EXAMPLE 1.3.
- *Activity: to look for a graphical system in order to represent natural numbers.*
- *Possible resulting mathematical knowledge: to write as many sticks as units are in the amount we want represent (*e.g. *IIIIII represents the number seven).*
- *Another possible resulting mathematical knowledge: the previous technique can be developed to get an additive system, which constitutes a new mathematical knowledge facing the same problematic activity.*

Mathematical knowledge can be organized in two levels:

- *praxis:* formed by *types of tasks*, plus the *techniques* to face these types of tasks.
- *logos:* the set of explanations intended to justify and support the validity of the techniques, to study the scope of the techniques,...

In fact, this two-levels organization turns out to be suitable for every human knowledge. We call *praxeology* a 3-tuple $(\{T_1, T_2, T_3, \dots\}, \tau; \theta)$ where T_i is a type of tasks, τ is a set of techniques to face the tasks, and θ is the corresponding logos.

Fundamental Postulate of the ATD: every human activity can be described in terms of praxeologies.

A *mathematical praxeology*(=MP) is a praxeology in which the techniques are of mathematical nature.

EXAMPLE 1.4.
- *Task: to subtract in our positional numeral system in basis 10.*
- *Technique: the borrowing algorithm (el algoritmo de "pedir prestado").*
- *Logos: using a very formal language, this can be summarized as follows*
  - $a \cdot 10^{n+1} + b \cdot 10^n = (a-1) \cdot 10^{n+1} + (10+b) \cdot 10^n$ *with* $1 \leq a, b \leq 9$.
  - *If* $1 \leq a < b \leq 9$, *then* $10 + a - b \leq 10 - 1 = 9$. *This ensures that, once the borrowing activity has ended, the result of every vertical subtraction we have to do (one for each power of 10) is a number between 0 and 9.*

EXAMPLE 1.5.
- *Task: to represent graphically the natural numbers.*
- *Technique: to write as many sticks as units (*e.g. *we write IIIIII for seven).*
- *Logos:* $\emptyset$.

EXAMPLE 1.6. • *Task: to represent graphically the natural numbers.*
- *Possible techniques: the one corresponding to the additive, hybrid or positional system, using basis b.*
- *Logos: we have to prove that every natural number x can be expressed as a sum of powers of b.*

EXAMPLE 1.7. • *Task: to calculate the gcd of two numbers.*
- *Technique: to find the decomposition of these numbers as a product of powers of prime numbers, and then the gcd will be the product of the common prime numbers to the smallest power.*
- *Logos: it has several complicated steps*
  1) *To prove that every natural number admit a decomposition as a product of prime numbers.*
  2) *To prove that the product of the common prime numbers to the smallest power does not depend on the decompositions. (Of course, this step woul be achieved if we were abel to prove that these kind of decompostions are always unique!)*
  3) *To prove that the product of the common primer numbers to the smallest power is the gcd.*

EXAMPLE 1.8. • *Task: to calculate the gcd of two numbers.*
- *Technique: to use repeatedly the Eculidean division until we get a division whose remainder is 0.*
- *Logos: in a division with divisor D, dividend d, quotient q and remainder r, one has that $gcd(D, d) = gcd(d, r)$. Of course, this fact would require, in turn, additional logos!*

To do Mathematics is to put into practice a MP, and *to study Mathematics* is to construct (in the case of the researcher) or to reconstruct (in the case of the sudent) a MP in order to face some problematic tasks.

2. Didactic moments

The study process can be described in terms of six *didactic dimensions* or *didactic moments*. Each dimension can be lived with a different intensity, in several moments, as many times as needed along the study process, and it is even usual the combination of several dimensions at the same time. Anyway, it is important to notice that:

- Each of these dimensions has a specific role, a specific purpose, necessary for the good development of the study process.
- There exists a global internal dynamic which appears in certain relationships between these dimensions.

Chevallard [**3**, pages 249–255], describes the six dimensions of the process of study of a MP, say O, as follows:

1) The *first didactic moment* is the *first meeting* with the mathematical praxeology O. Such a first meeting can happen in different ways, but ideally, if one wants to stress the functional character of mathematics, this would be through the types of tasks. Unfortunately, it is rather frequent to have a first meeting with a mathematical praxeology via its logos part.
2) The *second moment* consits of the *exploration* of the types of tasks T_i and the *construction of techniques* suitable for this types of tasks.
3) The *third moment* is the corresponding *setting-up of the logos enviroment.*
4) The *fourth moment* is the *work of the technique*, which should improve this technique transforming it into a more powerful tool.
5) The *fifth moment* is the *institutionalization.* Here we have to make precise, after having been working for while in O, which are the definitive constituents conforming O. For example, it is probably that we'll have to rule out some weak techniques.
6) The *sixth moment* is the *evaluation*, closely related to the institutionalization. In practice, one always has to look at what has been learned, to evaluate to what extent our purposes have been fulfilled,...

3. Epistemological Reference Model

3.1. Definition.

DEFINITION 3.2. *An* Epistemological Reference Model(=ERM) *is made of:*

1) *A problematic initial question* q*, which involves one or more types of tasks.*
2) *A tree of MP in which:*
   2.1) *each MP can add new types of tasks to the set of types of tasks involved in* q*,*
   2.2) *each MP is a (perhaps partial) answer to* q *and to the new types of tasks created in a previous MP (in case there is a previous one),*
   2.3) *each MP appears as a development of a previous MP (in case there is a previous one) due to the limitations of this one to provide asnwers to some aspects of the types of tasks under consideration.*

REMARK 3.3. *In a ERM, every MP appears as a solution to a problem, and so its functional or practical character is underlined.*

An ERM allows:

- the design, the management and the analysis of a study process,
- to analyse the spontaneous 'epistemology' of a teacher, which usually reflects the dominant epistemological model in a scholar institution.

3.4. Example. Let us summarize the ERM about numeral systems developed in [**4**].

The initial question q is: *How to write natural numbers in a way useful for the development of elementary aritmetic?*

We can rephrase the initial question as a list of types of tasks:

T_1) To express natural numbers using symbols avoiding any ambiguity.
T_2) To express natural numbers using a small amount of symbols.
T_3) To express each natural numbers using a not very large chain of symbols.
T_4) To express natural numbers in such a way that comparaison is easy.
T_5) To represent natural numbers in such a way that it is possible to develope a reliable and economic algorithm for the addition.
T_6) Idem for the subtraction.
T_7) Idem for the multiplication.
T_8) Idem for the Euclidean division.
T_9) Idem for the calculus of divisors and multiples.

As an answer to this list of types of tasks we have consider a tree of MP:

$$\begin{array}{ccccccc} MP_i & \longrightarrow & MP_a & \longrightarrow & MP_h & \longrightarrow & MP_p \\ & & \downarrow & & & & \\ & & MP_r & & & & \end{array}$$

where MP_i corresponds to the initial numeral system, MP_a corresponds to the additive numeral system (for example the Egyptian), MP_r corresponds to the Roman numeral system, MP_h corresponds to the hybrid numeral system (for example the Chinese) and MP_p corresponds to the positional numeral system.

- Each MP had techniques allowing to face the types of tasks T_1 , ... , T_9 to some extent.
- Each technique had its logos (except for those that, given its extreme simplicity, did not required a serious explanation, for example the technique facing T_1 in MP_i).

- The step from one MP to the following has been motivated for the quest of new better techniques.

3.5. ERM and didactic moments. An ERM allows to design a study process which pays attention to the sixt didactic moments:

- First meeting moment: starting with a ERM to design a study process allows a first meeting with the types of tasks to which the forthcoming techniques are answers. This emphasizes the functionality of mathematics.
- Exploration and elaboration of the technique: from the moment in which we face the tasks of q we start a quest of an asnwer.
- Setting-up of the logos enviroment: we look for an explanation of the technique.
- Work of the technique: we test the technique with several examples.
- Institucionalization and evaluation: we evaluate what we have done and we decide the elements (tasks, techniques, justifications) we keep and which are the elements we cast aside. The evaluation can be done under different criteria: pertinence of a tasks, scope or reliability or economy of a technique,… After the institucionalization and the evaluation we decide to go from one MP to another in the praxeological tree.

Remark 3.6. *Sometimes the justitifcation of some technique τ can be regarded as a new task T_*. Thus, the technique τ_* developed to face T_* becomes a part of the logos corresponding to τ.*

4. Glossary

- *to add* = sumar
- *addend* = sumando
- *addition* = suma
- *additive* = aditivo/a
- *didactic* es un adjetivo en Inglés, y no un sustantivo. Así, no podemos traducir "Didáctica de las matemáticas" como "Didactic of Mathematics". La expresión que se usa en Inglés en lugar de "Didáctica de las matemáticas" es "Mathematics Education".
- *to divide* = dividir
- *dividend* = dividendo
- *divisor* = divisor
- *great common divisor (gcd)* = máximo común divisor (mcd)
- *hybrid* = híbrido
- *least common multiple (lcm)* = mínimo común múltiplo (mcm)
- *mathematical* = matemático/a
- *minuend* = minuendo

- *multiple* = múltiplo
- *to multiply* = multiplicar
- *numeral system* = sistema de numeración
- *praxeology* = praxeología
- *quotient* = cociente
- *remainder* = resto
- *to substract* = restar
- *substraction* = rest
- *subtrahend* = sustraendo
- *task* = tarea
- *technique* = técnica