Mathematics and Mathematics Education I: magnitudes and measurement

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Contents

CHAPTER 3

Measurement and magnitudes

1. Magnitudes: amount, measurement, unit and order

A *magnitude* is an attribute that can vary in amount, in a quantitative way. The *amounts of magnitude* are the values of these atributes. Thus, *to measure* an amount of magnitude is to determine the proportion of this amount with respect to a fixed amount taken as a reference, the so-called *unit of measurement*. Two objects have the same *order* according to a certain common magnitude (we speak of *order of magnitude*) if it is reasonable to measure their amount of magnitud by using the same unit of measurement.

2. Situations of measurement

There are two tipical situations in which measurement of magnitudes is involved:

1) To *communicate* to other people (far in space or in time) how many things we have, or which is the size of some object, or how the amounts change as a consequence of certain transformations. The impossibility of transporting some colections of objects in space or in time forces us to take a transportable object (the unit of measurement) which is then taken as a reference.
2) To *look for relations* between amounts of two or more magnitudes. This activity characterizes the experimental scientific work.

3. Precision and measuring errors

The *precision* of a measuring instrument is the minimal variation of amount of magnitude that can be determined without error. The preciser an instrument is, the bigger is the number of significant figures that can be obtained with it.

EXAMPLE 3.1. *Imagine a meter in which all the milimeters appear. Then this meter has precision of one milimeter, since in the measurements we do with it we are not able to detect precisely differences of less than a milimeter.*

Assume we have a set of measurements $x_1, \ldots, x_n$ of amounts of a certain magnitude of an object. The *average value*, $\overline{x}$, of this set of measurements is

$$\overline{x} = (x_1 + \cdots + x_n)/n.$$

The *absolute error* $e_a(x_i)$ of one of this measurements x_i is:

$$e_a = x_i - \overline{x}.$$

The *relative error* $e_r(x_i)$ of one of this measurements x_i is:

$$e_r(x_i) = e_a(x_i)/\overline{x}.$$

The *dispersion error* e_d is:

$$e_d = (e_a(x_1) + \cdots + e_a(x_n))/n.$$

After several measurements and the calculation of the dispersion error, the *result of the measurement* must be expressed as the average value plus/minus the dispersion error:

$$\overline{x} + / - e_d.$$

EXERCISE 3.2. *Nine students have stimate the mass (generally refered to as 'weight') of an object. They have obtained the following weights (in kilograms):*

$$6.2\ ,\ 6.3\ ,\ 6.0\ ,\ 6.2\ ,\ 6.1\ ,\ 6.5\ ,\ 6.2\ ,\ 6.1\ ,\ 6.2$$

Determine the absolute and relative error of each measurement, and the dispersion error. Write the result of the measurement.

4. Measuring systems: regular/irregular, private/public

A measuring system is *irregular* if it uses units of measurement of different nature.

EXAMPLE 4.1. *Using sheets of papers, pencils and forks in order to measure lengths.*

This is problematic to calculate (how many forks fit in the length of a sheet of paper?). Thus, it is convenient to use a *regular* system.

EXAMPLE 4.2. *To measure lengths, one could use a sheet of paper, then half of the sheet, then the quarter of the sheet, and so on.*

It is easy to imagine good reasons for the use of a measuring system universally accepted. These measuring systems have the name of *legal*, as its use is regulated by laws.

The *Metric System* is our regular and legal measuring system. Is a decimalised measuring system, in the sense that the change from units to subunits, and vice versa, is done in tens in *linear* magnitudes (*e.g.* length), and in powers of ten in *multilinear* magnitudes (*e.g.* area, volume).

5. Fundamental or linear magnitudes vs. derivated or multilinear magnitudes

Once the unit of measurement is defined for certain magnitudes, one can define from them the corresponding units for other magnitudes. The first magnitudes are said to be *fundamental* (for example, length, time,. . .), and the second magnitudes are said to be *derivated* (for example, speed). When a magnitude is derivated from a single fundamental one and the unit of measurement of the derivated magnitude is 'several times' the unit of the fundamental one, then we say the fundamental one is *linear* (for example, length) and the derivated is *multilinear* (for example, area or volume).

REMARK 5.1. *The fact of being fundamental or derivated is not intrisic to the magnitude. A measuring system has to set precisely which are the fundamental magnitudes from which any other magnitude is to be derivated.*

6. International System of Units (SI)

6.1. Tables. This is the name adopted in the XI General conference of weights and measures (which took place in Paris in 1960) to set a universal measuring system, based on the mks (meter-kilogram-second) system. In this conference six fundamental and two complementary magnitudes where defined. In 1971 another fundamental magnitude was added, the *mole*. See the following table:

Magnitudes	**Name of the basic unit**	**Symbol**
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Amount of a chemical substance	Mole	mol
Luminous intensity	Candela	cd
Complementary magnitudes		
Planar angle	Radian	rad
Solid angle	Steradian	sr

In the following tabel we have multiples and submultiples:

10^n	**Prefix**	**Symbol**
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^2	hecto	h
10	deca	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	mili	m
10^{-6}	micro	μ
10^{-9}	nano	n

6.2. Metro. This unit was created by the French Academy of Sciences in 1791. It was originally defined as one ten-millionth of the distance from the Earth's equator to the North Pole (at sea level). Since 1983, it has been defined as *the length of the path travelled by light in vacuum during a time interval of 1/299792458 of a second.*

6.3. Second. For centuries, the time has been measured around the world using the rotation of the Earth. The second, the unit of time, was defined as 1/86400 of a mean solar day. However, the Earth rotation is not constant enough to be used as a reference to measure time. Thus, in 1967 the second was redefined using properties of the caesium atom: *the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.*

6.4. Kilogram. Is the only unit of measurement of the SI which is still defined by using a reference object, the so-called *International Prototype of the Kilogram*, instead of a fundamental physical property. This object is in the custody of the International Bureau for Weights and Measures (BIPM) who hold it on behalf of the General Conference on Weights and Measures (CGPM). After the International Prototype Kilogram had been found to vary in mass over time, the International Committee for Weights and Measures (CIPM) recommended in 2005 that the kilogram be redefined in terms of a fundamental constant of nature. At its 2011 meeting, the General Conference on Weights and Measures (CGPM) agreed in principle that the kilogram should be redefined in terms of the Planck constant, but deferred a final decision until its next meeting, scheduled for 2014.

7. Relationship between different magnitudes

7.1. Mass and weigth. The *mass* is the quantity of matter in an object. More specifically, *inertial mass* is a quantitative measure of an object's resistance to acceleration. In addition to this, *gravitational* *mass* is a quantitative measure that is proportional to the magnitude of the *gravitational force* which is

- exerted by an object (*active gravitational mass*), or
- experienced by an object (*passive gravitational force*)

when interacting with a second object.

Weight is the gravitational force acting on a given body –which differs depending on the gravitational pull of the opposing body (e.g., a person's weight on Earth vs on the Moon)– while mass is an intrinsic property of that body that never changes.

In other words, an object's weight depends on its environment, while its mass does not. On the surface of the Earth, an object with a mass of 50 kilograms weighs 491 newtons; on the surface of the Moon, the same object still has a mass of 50 kilograms but weighs only 81.5 newtons. Restated in mathematical terms, on the surface of the Earth, the weight W of an object is related to its mass m by W = mg, where g = 9.80665 m/s2 is the Earth's gravitational field, (expressed as the acceleration experienced by a free-falling object).

The identification of mass with weight at a popular level is very big. In schoolar practice it is very difficult to distinguish these two magnitudes. Moreover, many familiar tools intended to measure mass (*e.g.* balance scales = balanzas) are in fact tools to measure weight. Thus, it is not recomended to distinghish these two magnitudes in Primary Education.

7.2. Volume and capacity. *Volume* is the quantity of three-dimensional space enclosed by some closed boundary, for example, the space that a substance or shape occupies. Volume is often quantified numerically using the SI derived unit, the *cubic metre* m^3.

Capacity of a container is the amount of fluid (gas or liquid) that the container could hold, rather than the amount of space the container itself displaces.

Capacity is usually quantified numerically using the liter, l. A liter is a non-SI metric system unit of volume corresponding to 1 cubic decimetre, dm^3. That is to say, the volume of a closed container with a capacity of 1 liter is 1 dm^3.

7.3. Area and surface. A *surface* is a geometric object (a set of points satisfying certain properties) but not a magnitude. We can speak of planar surfaces, warped surfaces,...

Area is a quantity that expresses the extent of a surface or shape, or planar lamina. The area of a shape can be measured by comparing the shape to squares of a fixed size. In the International System of Units (SI), the standard unit of area is the *square metre* (written as m^2), which is the area of a square whose sides are one metre long.

8. Glossary

- *absolute error* = error absoluto
- *acceleration* = aceleración
- *active gravitational mass* = masa gravitacional activa
- *amount* = cantidad
- *amount of a chemical substance* = cantidad de sustancia química
- *ampere* = amperio
- *area* = área
- *attribute* = atributo
- *average value* = valor medio
- *candela* = candela
- *capacity* = capacidad
- *cubic* = cúbico/a
- *dispersion error* = error de dispersión
- *electric current* = corriente eléctrica
- *to exert* = ejercer
- *gravitational force* = fuerza gravitacional
- *inertial mass* = masa inercial
- *kilogram* = kilogramo
- *length* = longitud
- *linear* = lineal
- *luminous intensity* = intensidad luminosa
- *magnitude* = magnitud
- *mass* = masa
- *to measure* = medir
- *measure* = medida
- *measurement* = medición
- *measuring instrument* = instrumento de medida
- *measuring system* = sistema de medida
- *meter* = metro
- *metric system* = sistema métrico
- *milimeter* = milímetro
- *mole* = mol
- *multilinear* = multilineal
- *order* = orden
- *order of magnitude* = orden de magnitud
- *passive gravitational force* = fuerza gravitacional pasiva
- *planar angle* = ángulo plano
- *precision* = precisión
- *quantitative* = cuantitativo
- *quantity* = cantidad
- *radian* = radian
- *relative error* = error relativo

- *second* = segundo (como unidad de tiempo y como número ordinal)
- *solid angle* = ángulo sólido
- *square* = cuadrado
- *steradian* = estereoradián
- *temperature* = temperatura
- *time* = tiempo
- *unit* = unidad
- *unit of measurement* = unidad de medida
- *volume* = volumen
- *warped* = alabeado/a
- *weight* = peso