Mathematics and Mathematics Education I: magnitudes and measurement

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Contents

CHAPTER 4

ERM about measurement of magnitudes

According to Brousseau [**1**] there are three domains linked to the measurement of magnitudes, namely:

1) The domain of the specific measurable objects.
2) The domain corresponding to the process of defining the measurement map.
3) The domain of the numerical structure.

Despite these three domains are frequently presented to the students as disconected, they are strongly related, as can be seen in the ERM proposed in [**4**].

1. Description of the ERM

1.1. Initial praxeologies: MP_0 y MP_1. We consider a system $S(G)$ of objects $\{a_i\}_{i\in I}$ (specific material or mathematical objects) which can be measured according to some magnitude G.

We consider the following tasks, which conform our initial question:

T_1) To compare (present or absent) amounts of magnitude.

T_2) To construct an amount of magnitude equivalent to a given (present or absent) amount.

The initial techniques[1] to face these tasks consist of direct manipulation of objects:

$\tau_{0,1}$) We can construct a new measurable object attaching two or more objects (binary operation denoted by $a_i \oplus a_j$, only defined when $a_i \neq a_j$).

$\tau_{0,2}$) We can do a naive comparison (via visual examination, using an instrument -for example, a balance scale-,...) of two different objects, a y b, with respect to G and decide whether they are equivalent (in this case we write $a \sim b$) or one is smaller than the other (we write $a \prec b$).

The logos underlying these techniques is the following:

$\theta_{0,1}$) The binary relation $\sim$ in $S(G)$ is symmetric and transitive, namely, it satisfies:

- if $a \sim b$ then $b \sim a$,

[1] Warning about numbering: the subindex i, j in the technique $\tau_{i,j}$ means that it is the jth technique we have considered in the ith praxeology.

– if $a \sim b$ and $b \sim c$ then $a \sim c$.

$\theta_{0,2}$) The binary relation $\prec$ in $S(G)$ satisfies:
– if $a \prec b$ and $b \prec c$ then $a \prec c$,
– if $a \prec b$ then we do not have that $b \prec a$,
– for every pair of objects a , $b \in S(G)$ we have that, if $a \nsim b$, then either $a \prec b$ or $b \prec a$.

$\theta_{0,3}$) There is a certain compatibility between $\oplus$, $\sim$ and $\prec$. For example,
– if $a \sim b$ then $a \oplus c \sim b \oplus c$,
– if $a \prec b$ then $a \oplus c \prec b \oplus c$.

Thus, the first praxeology would be:

$$MP_0 = (\{T_1, T_2\}, \{\tau_{0,1}, \tau_{0,2}\}; \{\theta_{0,1}, \theta_{0,2}, \theta_{0,3}, ...\})$$

Example 1.2.
- *G=length.*
- *S(G)= several cardboard bands of different colors.*
- *$\sim$: two bands are equivalent if we check they have the same amount of length after putting them together.*
- *$\prec$: we have $a \prec b$ if after putting them together we check that a is shorter than b.*
- *$\oplus$= juxtaposition of bands, with coincidental tails.*

Example 1.3.
- *G=weight.*
- *S(G)= several objects with different amounts of weight, allowing the use of a scale.*
- *$\sim$: two objects are equivalent if they balance the scale.*
- *$\prec$: we have $a \prec b$ if, after putting a and b in the two pans of a balance scale, the pan of a falls more than the pan of b.*
- *$\oplus$= we put both objects in the same pan of the scale.*

We can see that the techniques based on direct handling have a small scope. For instance, the fact that one object can not be attached to itself might be a problem in order to check whether one object has twice the amount of magnitude of another one. It is also a problem the imposibility of handling a big set of objects (for example, because they do not fit all together in the pan of the scale).

We can find a solution to these problems after delving in the logos of the previous praxeology. The following would be part of the following logos:

$\theta_{1,1}$) When two objects a and b satisfy that $a \sim b$ we can consider they to be 'the same' object to some extent. Formally, this corresponds to go from $S(G)$ to the quotient set $MS(G) = S(G)/\sim$. Note that in this new praxeology we can write

$$\text{green band} = \text{blue band},$$

not meaning that they are the same band but that they have the same amount of length.

This allows new techniques:

$\tau_{1,1}$) We go from $\oplus$ to a symbolic operation denoted by $+$. It is symbolic in the sense that it is made not with real fisical objects but with writings symbolizing fisical objects. Anyway, these writings can be names of fisical objects, *e.g.*

$$\text{green band} + \text{blue band} + \text{blue band}.$$

$\tau_{1,2}$) We go from $\prec$ to a symbolic operation denoted by $\leq$. Again, it is a symbolic operation in the sense that it is not applied to fisical objects but with writings. For example, we can write

$$\text{green band} \leq \text{blue band}.$$

This means that the amount of length of the green band is smaller or equal to the amount of length of the blue one.

EXAMPLE 1.4. *A student do the step from MP_0 to MP_1 when she stops to confine herself in the realm of fisical objects and starts to handle symbols. For example, assume we are studying the magnitude length using a set of cardboard bands:*

$$S(\textit{length}) = \{\textit{cardboard bands}\}.$$

Let b_1 , b_2 and b_3 be different cardboard bands. In $S(\textit{length})$ I can not attach b_3 to itself 5 times, however in $MS(\textit{longitud})$ I do can write

$$b_1 = b_2 + 5 \cdot b_3$$

to say that the amount of length of b_1 equals the amount of length of b_2 plus 5 times the amount of length of b_3.

Let us make a little bit explicit the logos of the new MP:

$\theta_{1,3}$) The new operation $+$ is defined in $MS(G)$ as follows:

$$[a] + [b] = [a \oplus b].$$

Note that, despite we could not add an object with itself in $S(G)$, that is to say, the operation $a \oplus a$ did not make sense, in $MS(G)$ we can add an object with itself, that is to say, we can do $[a] + [a]$. Indeed, we can do

$$[a] + [a] = [a \oplus b]$$

where $b \sim a$. Thus,

$$\text{green band} + \text{blue band} + \text{blue band}$$

is meaningful, since it might mean that we attach a blue band to a green band, and then we attach another band with the same amount of length that the blue one.

$\theta_{1,4}$) The new operation $\leq$ is defined in $MS(G)$ as follows:

$$[a] \leq [b] \text{ if and only if } \begin{cases} a \prec b \text{ and so we can write } [a] \lneq [b] \\ a \sim b \text{ and so we can write } [a] = [b]. \end{cases}$$

This operation is a *total order*, *i.e.* it satisfies the following properties:

- Reflexivity: $[a] \leq [a]$.
- Antisymmetry: if $[a] \leq [b]$ and $[b] \leq [a]$ then $[a] = [b]$.
- Tansitivity: if $[a] \leq [b]$ and $[b] \leq [c]$ then $[a] \leq [c]$.
- Total: for any two symbols $[a]$ and $[b]$ we have either $[a] \leq [b]$ or $[b] \leq [a]$.

$\theta_{1,5}$) Compatibility between $+$ and $\leq$: if $[a] \leq [b]$ then $[a] + [c] \leq [b] + [c]$.

Thus, we have move from MP_0 to a new praxeology which results from developing the logos of MP_0 and enlarging the set of techniques and the logos of MP_0:

$$MP_1 = (\{T_1, T_2\}, \{\tau_{0,1}, \tau_{0,2}, \tau_{1,1}, \tau_{1,2}\}; \{\theta_{0,1}, \theta_{0,2}, \theta_{0,3}, ...\} \cup \{\theta_{1,i}\}_{i=1}^{5})$$

1.5. A transition praxeology: MP_2**.** Problems of MP_0 and MP_1:

1) The techniques $\tau_{1,1}$, $\tau_{1,2}$, consisting in the working with symbols according to some rules explicited by the logos $\theta_{1,i}$, $1 \leq i \leq 5$, could not be suitable when doing comparisons.
2) We still do not have a technique allowing to communicate a faraway receiver an amount of magnitude effectively. For example, I can say "Construct a cardboard band with the same amount of length that my yellow band plus my blue band." But if the receiver doesn't know whether any of her bands has the same amount of length that my yellow band, then she could not profit my message.

To solve these difficulties we consider a new type of tasks:

T_3) To find a not too big family of symbols $\{u_i\}_{i\in I}$ in $MS(G)$ allowing to write most of the symbols of $MS(G)$ as a linear combination of the u_i with natural coefficients:

$$\Sigma_{i\in I}\lambda_i u_i \ , \ \lambda_i \in \mathbf{N}.$$

The symbols u_i will be said to be *generators* of $MS(G)$.

The solution of this type of tasks implies a solution to the problems (1) and (2) mentioned before. Indeed:

1) The comparison is simpler, since we have

$$\Sigma_{i\in I}\lambda_i u_i \leq \Sigma_{i\in I}\mu_i u_i$$

when

$$\lambda_i \leq \mu_i \text{ for every } i \in I.$$

2) The communication of an amount of magnitude to a receiver is also simpler since to communicate

$$\Sigma_{i\in I}\lambda_i u_i$$

it suffices to communicate the numbers λ_i once the receiver knows which are the generators.

In relation to T_3 we have more types of tasks:

T_4) Given a family of generators, can we still **reduce** the number of generators?
T_5) Given a family of generators, can we express **every** symbol as a linear combination of the generators? If not, which are the nearest linear combinations?
T_6) Can the symbols of $MS(G)$ be written **in a unique way** as a linear combination of the generators?
T_7) Given two families of generators, $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$, for example one owned by a transmitter and another one owned by a receiver, how can we express in terms of the $\{v_j\}_{j\in J}$ the messages written in terms of $\{u_i\}_{i\in I}$?

The suitable techniques to face $T_3 - T_7$ are:

$\tau_{2,1}$) To solve T_7 it suffices to know how to express each u_i in terms of the v_j.
$\tau_{2,2}$) To face T_4 and T_6 it is useful to have linearly independent generators. Thus, if for exampe $u_j = 2\cdot u_i$, it is better not to include u_j as a generator.
$\tau_{2,3}$) Another way of facing T_4, and at the same time T_7, is to enlarge the set of numbers we use as coefficients. For example, if $3\cdot u_j = 2\cdot u_i$, we can keep u_i in our set of generators, and eliminate u_j, using in turn the rational number $\frac{2}{3}$ as a coefficient. Also, if a symbol can not be expressed in terms of $\{u_i\}_{i\in I}$ if we only use natural numbers as coefficients, perhaps this symbols can be expressed in terms of the generators if we allow ourselves to use rational coefficients. Moreover, to enlarge the set of numbers used as coefficients is useful to tackle T_5, since we also enlarge the set of measurable objects.
$\tau_{2,4}$) As we enlarge the set of numbers we use as coefficients

$$\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{K}' \subseteq \ldots$$

we have to develope techniques to transform expressions or relations with coefficients in $\mathbb{K}$ into expressions or relations with coeficientes en $\mathbb{K}'$. Por example,

$$3\cdot u_j = 2\cdot u_i \Leftrightarrow u_j = \frac{2}{3}\cdot u_i.$$

Note that in this MP the *measurement map* appears for the first time:

$$MS(G) \subseteq \langle u_1 , u_2 , \dots , u_s \rangle \to \mathbb{K}^s , \Sigma_{i=1}^s \lambda_i u_i \mapsto (\lambda_1 , \dots , \lambda_s)$$

This map enables to conect the domain of specific measurable objects with the domain of the numerical structure.

1.6. Final praxeology: MP_3**.** The MP_2 gives rise to the following techniques which solve all the tasks under consideration:

$\tau_{3,1}$) Fix a single generator $\{u\}$ and enlarge the set of numbers used as coefficients to get the whole set of positive rational numbers $\mathbb{Q}^+$. We try to choose a generator u with an *order of magnitude* (see § 1) suitable for the objects we tipically deal with.

$\tau_{3,2}$) In fact, instead of considering all the positive rational numbers we just consider the coefficients of the form

$$a \cdot 10^m , m \in \mathbb{Z},$$

and we give a proper name to the symbols of the form $10^m \cdot u$, $m \in \mathbb{Z}$. This is directly related to the fact that our numeral system is a positional one of basis 10. Thus, if u is one meter we give names (. . . , milimeter, centimeter, decameter, hectometer,. . .) to the symbols of the form

$$10^m \cdot u.$$

2. Glossary

- *antisymmetry* = antisimetría
- *attach* = adjuntar
- *band* = banda
- *binary* = binario/a
- *cardboard* = cartulina
- *coefficient* = coeficiente
- *to compare* = comparar
- *comparison* = comparación
- *domain* = dominio
- *to enlarge* = ampliar
- *generator* = generador
- *to handle* = manejar
- *magnitude* = magnitud
- *map* = aplicación
- *measurable* = medible
- *measurement* = medición
- *order of magnitud* = orden de magnitud
- *rational number* = número racional
- *receiver* = receptor
- *reflexive* = reflexivo/a

- *reflexivity* = reflexividad
- *set* = conjunto
- *symmetric* = simétrico/a
- *symmetry* = simetría
- *transitive* = transitivo/a
- *transitivity* = transitividad