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ALL MAPS OF TYPE 2^∞ ARE BOUNDARY MAPS

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Abstract. Let f be a continuous map of an interval into itself having periodic points of period 2^n for all $n \geq 0$ and no other periods. It is shown that every neighborhood of f contains a map g such that the set of periods of the periodic points of g is finite. This answers a question posed by L. S. Block and W. A. Coppel.

1. Introduction

Let I be a real compact interval and $C(I,I)$ be the metric space of continuous maps of I into itself with the distance between two elements f,g defined by $\varrho(f,g) = \max\{|f(x)-g(x)| : x \in I\}$. Let $\mathbb{N}$ be the set of positive integers. A point $p \in I$ is a *periodic point* of a map f if $f^n(p) = p$ for some $n \in \mathbb{N}$. The *period* of p is the least such integer n, and the *orbit* of p under f is the set $\{f^k(p) : k = 0,1,\dots,n-1\}$. We refer to such an orbit as a *periodic orbit* of f of period n.

A map $f \in C(I,I)$ is *piecewise monotone* if there are points $\min I = a_0 < a_1 < \cdots < a_n = \max I$ such that for every $k \in \{1,2,\dots,n\}$, the restriction of f to the interval $[a_{k-1},a_k]$ is (not necessarily strictly) monotone. When speaking of a piecewise monotone map we can always take n as the minimal positive integer with this property and call the points $a_1,\dots,a_{n-1}$ *turning points* of f (though they still are not uniquely determined by f).

Consider the Sharkovskii ordering of the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \cdots \succ 4\cdot 3 \succ 4\cdot 5 \succ 4\cdot 7 \succ \cdots \succ \cdots$$

$$\succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ \cdots \succ \cdots \succ 2^\infty \succ \cdots \succ 2^n \succ \cdots \succ 4 \succ 2 \succ 1.$$

We will also use the symbol $\succeq$ in the natural way. For $t \in \mathbb{N} \cup \{2^\infty\}$ we denote by $S(t)$ the set $\{k \in \mathbb{N} : t \succeq k\}$ ($S(2^\infty)$ stands for the set $\{1,2,4,\dots,2^k,\dots\}$). Let $f \in C(I,I)$ and $\mathrm{Per}(f)$ be the set of periods of its periodic points.

Sharkovskii's Theorem ([Sh1],[Sh2]). *For every $f \in C(I,I)$ there exists a $t \in \mathbb{N} \cup \{2^\infty\}$ with $\mathrm{Per}(f) = S(t)$. On the other hand, for every $t \in \mathbb{N} \cup \{2^\infty\}$ there exists an $f \in C(I,I)$ with $\mathrm{Per}(f) = S(t)$.*

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If $\mathrm{Per}(f) = S(t)$, then f is said to be *of type* t. When speaking of types we consider them to be ordered by the Sharkovskii ordering. So if a map f is of type 2^∞ or greater than 2^∞ or less than 2^∞, then, respectively, $\mathrm{Per}(f) = \{1, 2, \dots, 2^k, \dots\}$ or f has a periodic point with period not a power of 2 or $\mathrm{Per}(f) = \{1, 2, \dots, 2^n\}$ for some $\mathbb{N}$. The set $\mathrm{Per}(f)$ is finite if and only if f is of type less than 2^∞. The topological entropy of f is positive if and only if f is of type greater than 2^∞ (see [BF] for the "if" part and [Mi] for the "only if" part or [ALM, Theorem 4.4.19]).

Recall that it is very easy to see that any neighborhood of any map f contains maps of types greater than 2^∞ (even maps of type 3) (see [Kl]). Contrary to the maps of types greater than 2^∞, the maps of types less than 2^∞ do not form a dense set in $C(I, I)$. In fact, this set is nowhere dense in $C(I, I)$. To see this use the following Block's Theorem.

Block's Theorem ([Bl]). *Let $f \in C(I, I)$ and let $n \in \mathrm{Per}(f)$. Then there exists a neighborhood $U(f, n)$ of f such that for all $g \in U(f, n)$ we have $\mathrm{Per}(g) \supset S(n) \setminus \{n\}$.*

So, if f is of type greater than 2^∞, then there is a neighborhood of f containing no map of type less than (or equal to) 2^∞ . L. S. Block and W. A. Coppel (see [BC], the end of chapter II.4) posed the question of whether any neighborhood of any map of type 2^∞ contains a map of type less than 2^∞ . We answer this question in the affirmative by proving the following

Theorem. *Let $f \in C(I, I)$ be of type 2^∞ . Then every neighborhood of the map f contains a piecewise monotone map of type less than 2^∞ .*

We prove this theorem in two steps. First we prove it under the additional assumption that f is piecewise monotone. Then we prove that in any neighborhood of a map of type 2^∞ there is a piecewise monotone map of type at most 2^∞ . But before going to the proof we wish to mention some aspects of this theorem.

Denote by G or E or L, respectively, the set of all maps of types greater than or equal to or less than 2^∞ . For any set A in the metric space $C(I, I)$ let $\mathrm{Bd}\, A$ denote the boundary of A. Our theorem and Block's Theorem show that $\mathrm{Bd}\, G = \mathrm{Bd}\, L = E \cup L$. This is what is meant by the title of this paper.

In the Sharkovskii ordering the smallest element is 1 and the largest one is 3. For any $n \in \mathbb{N} \setminus \{1\}$ denote by $\nu(n)$ the predecessor of n, i.e., the maximum (in the Sharkovskii ordering) of the set $S(n) \setminus \{n\}$. If f is of type $n \in \mathbb{N} \setminus \{1\}$ then, by Block's Theorem, there is a neighborhood U of f such that for every $g \in U$, the type of g is at least $\nu(n)$. It is also known (and not difficult to show) that for any $n \in \mathbb{N}$ there exist a map f_n of type n and a neighborhood U of f_n such that for every $g \in U$, the type of g is at least n. Our theorem shows that this is not true for $n = 2^\infty$. In other words, if f is of type 2^∞, and if for every $n \in \mathbb{N}$ the neighborhood U_n of f satisfies that every map in U_n is at least of type 2^n (the existence of such neighborhoods follows from Block's Theorem), then $\bigcap_{n=0}^\infty U_n = \{f\}$.

The Sharkovskii Theorem holds also for the set $C(\mathbb{R}, \mathbb{R})$ of continuous maps from the real line $\mathbb{R}$ into itself (in this case we have the additional possibility $\mathrm{Per}(f) = \emptyset$) (see, e.g., [ALM, Corollary 2.1.2]). It is known that Block's Theorem works also for maps from $C(\mathbb{R}, \mathbb{R})$ (now $\varrho(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$ may be infinite) (see [ALM], Remark 2.8.5). So, it is natural to ask whether at least the weaker form of the theorem without the words "piecewise monotone" is true in $C(\mathbb{R}, \mathbb{R})$. The answer is negative. In fact, for any $n \in \mathbb{N}$ take a map $\varphi_n \in C([0, 1], [0, 1])$ of type 2^n with $\varphi_n([0, 1]) \subset [1/4, 3/4]$ and $0 < \varepsilon_n < 1/4$ such that the ball $B(\varphi_n, \varepsilon_n)$

contains only maps with types at least 2^{n-1}. Then there are $a_n > 0$ and a map $\alpha_n \in C([0,a_n],[0,a_n])$ of type 2^n with $\alpha_n([0,a_n]) \subset [1, a_n - 1]$ such that the ball $B(\alpha_n, 1)$ contains only maps with types at least 2^{n-1} (take $a_n = 1/\varepsilon_n$ and $\alpha_n = h_n \circ \varphi_n \circ h_n^{-1}$ where h_n is the increasing affine map of $[0,1]$ onto $[0,a_n]$). Further, put $b_0 = 0$, $b_1 = 1$ and $b_{2n} = n + \sum_{i=1}^n a_i$, $b_{2n+1} = b_{2n} + 1$ for $n = 1, 2, \dots$ and let $\beta_n \in C([b_{2n-1}, b_{2n}], [b_{2n-1}, b_{2n}])$ be defined by $\beta_n = k_n \circ \alpha_n \circ k_n^{-1}$ where k_n is the increasing affine map from $[0,a_n]$ onto $[b_{2n-1}, b_{2n}]$. Then $f \in C(\mathbb{R},\mathbb{R})$ defined by

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ \beta_n(x), & \text{if } x \in [b_{2n-1}, b_{2n}], \\ \text{affine}, & \text{if } x \in [b_{2n-2}, b_{2n-1}] \end{cases}$$

is a map from $C(\mathbb{R},\mathbb{R})$ of type 2^∞ such that all maps from the ball $B(f,1)$ have types at least 2^∞ (use that $g([b_{2n-1}, b_{2n}]) \subset [b_{2n-1}, b_{2n}]$ whenever $g \in B(f,1)$).

2. Proof of the Theorem

If $A \subset I$, then $\operatorname{int} A$ or $\operatorname{diam} A$ will denote the interior or the diameter of A, respectively. If $J_1, J_2 \subset I$ are intervals, then $J_1 < J_2$ will mean that $\sup J_1 < \inf J_2$. If f is a map and A is a set, then $f|_A$ is the restriction of f to A. Let $f \in C(I,I)$. We say that $x \in I$ is *eventually periodic* if $f^n(x)$ is periodic for some n.

Given $f \in C(I,I)$, a closed subinterval J of I is periodic of period n if $f^n(J) = J$ and $f^k(J) \cap f^l(J) = \emptyset$ for any $0 \le k < l < n$. Further, we say that $S \subset I$ is a *(simple) solenoid* of f if $S = \bigcap_{n=0}^\infty \bigcup_{k=0}^{2^n-1} f^k(I^n)$ where for any n, I^n is a periodic interval of period 2^n such that $I^n \supset I^{n+1}$. The equality $S = \bigcap_{n=0}^\infty \bigcup_{k=0}^{2^n-1} f^k(I^n)$ will be said to be a *standard representation* of the solenoid S. Clearly, S is a compact set, $f(S) = S$ and S cannot contain any eventually periodic point of f.

Lemma 1. *Let $f \in C(I,I)$ and let R, S be different solenoids of f. Then $R \cap S = \emptyset$.*

Proof. This follows immediately from the proofs of Proposition 2.2 (7) and Lemma 2.15 in [Pr] (these proofs work without the assumption of piecewise monotonicity). ☐

Lemma 2. *Let $f \in C(I,I)$ be piecewise monotone and S be a solenoid of f. Then S must contain a turning point. In particular the number of solenoids of f is finite.*

Proof. If $S = \bigcap_{n=0}^\infty \bigcup_{k=0}^{2^n-1} f^k(I^n)$ is a standard representation of a solenoid S of f, then, taking any n, f cannot be monotone on all intervals $f^k(I^n)$, $k = 0, 1, \dots, 2^n - 1$. Consequently, there is at least one of the finitely many turning points of f belonging to S. The rest of the lemma follows from Lemma 1. ☐

The set of all limit points of the trajectory $(f^n(x))_{n=0}^\infty$ of a point x is called the *ω-limit set* of x under f and is denoted by $\omega_f(x)$. A standard well-known result says that every finite ω-limit set is a periodic orbit.

A well-known result implicit in several of Sharkovskii's papers and proved in [Sm] (see also [FS]) states that every infinite ω-limit set of a map of type 2^∞ is contained in a solenoid. In [Ge] it is proved that if a map f of type 2^∞ is additionally piecewise monotone, then any infinite ω-limit set of f is Cantor-like. Finally, we will substantially use a result from [JS] saying that every piecewise monotone map of type 2^∞ must have an infinite (Cantor-like) ω-limit set. Combining these facts we get

Lemma 3. *Every piecewise monotone map $f \in C(I, I)$ of type 2^∞ has infinite ω-limit sets, each of them being Cantor-like and contained in a solenoid.*

Now we are able to prove our main result under the additional assumption that f is piecewise monotone.

Lemma 4. *Let $f \in C(I, I)$ be of type 2^∞ and piecewise monotone. Then every neighborhood of f contains a piecewise monotone map of type less than 2^∞ .*

Proof. Let $\varepsilon > 0$. We wish to find a map $g \in C(I, I)$ of type less than 2^∞ with $\varrho(f, g) < \varepsilon$. Take $\delta > 0$ such that $\operatorname{diam} f(J) < \varepsilon$ whenever $\operatorname{diam} J < \delta$.

Special case. Assume that the map f has only one solenoid S (see Lemma 3 and Lemma 2).

In the standard representation $S = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} I_k^n$, $I_k^n = f^k(I^n)$, we may assume that $I_0^n \supset I_0^{n+1}$ for every n. Then $I_k^{n+1}, I_{k+2^n}^{n+1} \subset I_k^n$ for every n and $k = 0, 1, \ldots, 2^n - 1$. Denote by K_k^n the (maximal) closed interval lying between $\operatorname{int} I_k^{n+1}$ and $\operatorname{int} I_{k+2^n}^{n+1}$. Further, for every n denote $I(n) = \{I_k^n : k = 0, 1, \ldots, 2^n - 1\}$, $K(n) = \{K_k^n : k = 0, 1, \ldots, 2^n - 1\}$ and $\bigcup I(n) = \bigcup_{k=0}^{2^n-1} I_k^n$, $\bigcup K(n) = \bigcup_{k=0}^{2^n-1} K_k^n$.

Realize the following:

(i) There is a (sufficiently large) l_1 such that $\bigcup I(l_1)$ contains only those turning points of f which belong to S. Then, for any $n \geq l_1$, $\operatorname{int}(\bigcup K(n))$ does not contain any turning point of f.

(ii) There is an l_2 such that $I(l_2)$ contains an interval with diameter less than δ.

Take $l = \max\{l_1, l_2\}$. Then $I(l)$ contains an interval, say I_0^l, with diameter less than δ. Further, f is monotone on each of the intervals belonging to $K(l)$.

We have $P = I_0^{l+1} \cup K_0^l \cup I_{2^l}^{l+1} \subset I_0^l$ where we may assume that $I_0^{l+1} < I_{2^l}^{l+1}$. Define $g \in C(I, I)$ by

$$g(x) = \begin{cases} f(x), & \text{for } x \in I \setminus (K_0^l \cup I_{2^l}^{l+1}), \\ f(\max I_{2^l}^{l+1}), & \text{for } x \in I_{2^l}^{l+1}, \\ \text{affine on } K_0^l. \end{cases}$$

The map g is piecewise monotone and since $\operatorname{diam} f(P) < \varepsilon$, $g(P) \subset f(P)$ and f coincides with g on $I \setminus P$, we have $\varrho(f, g) < \varepsilon$. To finish the proof of the Special case, it is sufficient to show that g is of type less than 2^∞ .

Take any $x \in I$. If the trajectory $(g^n(x))_{n=0}^\infty$ does not visit $K_0^l \cup I_{2^l}^{l+1}$, then it coincides with the trajectory $(f^n(x))_{n=0}^\infty$, whence $\omega_g(x) = \omega_f(x)$. Since the set $\omega_g(x) = \omega_f(x)$ is not a subset of S (otherwise the trajectory $(g^n(x))_{n=0}^\infty$ would intersect $I_{2^l}^{l+1}$), it is a finite set, a periodic orbit of f (as well as of g) of period a power of 2.

Now suppose that $(g^n(x))_{n=0}^\infty$ visits the set $I_{2^l}^{l+1}$. Then the point x is eventually periodic with period 2^{l+1} and so $\omega_g(x)$ is finite.

Finally suppose that $(g^n(x))_{n=0}^\infty$ does not visit $I_{2^l}^{l+1}$ and, for some n_0, $g^{n_0}(x) \in K_0^l$. Then $(g^n(x))_{n=n_0}^\infty$ lies in $\bigcup K(l)$. Due to the fact that g is monotone on each of the intervals belonging to $K(l)$, for any $k \in \{0, 1, \ldots, 2^l - 1\}$ the set $M_k^l = \{y \in K_k^l : (g^n(y))_{n=0}^\infty$ lies in $\bigcup K(l)\}$ is an interval (possibly degenerate), the map $h = g^{2^l}$ maps this interval into itself and is monotone on it. Hence, for any $y \in M_k^l$ the cardinality of $\omega_h(y)$ is at most 2 (see [Co] or [Sh2]). Therefore, for our point x we get that $\omega_g(x)$ has cardinality 2^l or 2^{l+1}.

We have proved that for every $x \in I$, $\omega_g(x)$ is finite and, moreover, has cardinality a power of 2. So g is of type at most 2^∞ . It cannot be of type 2^∞ since otherwise, being piecewise monotone, by [JS] (cf. Lemma 3) it would have an infinite ω-limit set. Therefore g is of type less than 2^∞ .

General case. The map f has the solenoids $S_1, S_2, \dots, S_r$ (and no other ones) for some $r \in \mathbb{N}$.

Since the solenoids S_i, $i = 1, 2, \dots, r$, are compact and mutually disjoint, each of them has a positive distance from the others. So, if $S_i = \bigcap_{n=0}^{\infty} \bigcup_{k=0}^{2^n-1} (I_i)_k^n$ is a standard representation of S_i, there exists an m such that the sets $\bigcup (I_i)(m) = \bigcup_{k=0}^{2^m-1} (I_i)_k^m$, $i = 1, 2, \dots, r$, are pairwise disjoint. Now the same procedure which was applied in the Special case to a map having one maximal solenoid S can be applied r-times to our map f having r solenoids $S_1, S_2, \dots, S_r$. As a result we get a piecewise monotone map g of type less than 2^∞ with $\varrho(f, g) < \varepsilon$. □

In what follows, $h(f)$ denotes the topological entropy of f (see [AKM] or [ALM] for the definition and properties). Here we just recall that $h(f) \in [0, +\infty]$.

Lemma 5. *Let X be a compact metric space, and let $T : X \to X$ be a continuous map. Suppose that $(U_n)_{n \geq 1}$ is a sequence of open subsets of X, and let $S : X \to X$ be a continuous map such that $S(x) = T(x)$ for all $x \in X \setminus \bigcup_{n=1}^{\infty} U_n$ and $S|_{U_n}$ is constant for all $n \geq 1$. Then $h(S) \leq h(T)$.*

Proof. Set $V := \bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} S^{-k}(U_n)$ and $C := X \setminus V$. Obviously $S(C) \subset C$ and $S|_C = T|_C$. An easy calculation shows that $\Omega(S) \cap V$ equals an at most countable union of periodic orbits ($\Omega(S) \cap V$ may be empty), where $\Omega(S)$ denotes the nonwandering set of S (see [W] for the definition and properties). The variational principle (see [W, Theorem 8.6 and Corollary 8.6.1] and [W, Theorem 6.15]) give

$$h(S) = \begin{cases} h(S|_C), & \text{if } C \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

In the first case $h(S|_C) = h(T|_C) \leq h(T)$, and in the second case $h(S) \leq h(T)$ is trivial. □

Remark. An easy consequence of Lemma 5 is the following fact. If $f, g \in C(I, I)$ coincide outside an open set G, and if g is constant on every connected component of G, then $h(g) \leq h(f)$.

Lemma 6. *Let $f \in C(I, I)$ and let $[u, v] \subset I$ be a closed interval with* $\max f([u, v]) \in \{f(u), f(v)\}$. *Then there is a map $g \in C(I, I)$ with the following properties:*

(1) $g(x) = f(x)$ whenever $x \notin (u, v)$,
(2) $g|_{[u,v]}$ is (not necessarily strictly) monotone, and
(3) $h(g) \leq h(f)$.

Proof. Assume that $\max f([u, v]) = f(v)$ (if the maximum is attained at the point u, the proof is analogous). Define the map $g|_{[u,v]}$ as the so-called rising sun function corresponding to the map $f|_{[u,v]}$ when a rising sun is on the x-axis at $-\infty$ (equivalently, pour water into the graph of $f|_{[u,v]}$ until you get all "holes in the ground" full of water). More precisely, put

$$g(x) = \begin{cases} f(x), & \text{if } x \in I \setminus [u, v], \\ \max\{f(t) : u \leq t \leq x\}, & \text{if } x \in [u, v]. \end{cases}$$

Then (1) and (2) hold trivially, the continuity of g follows from the assumption that $\max f([u,v]) = f(v)$. Finally, the set $G = \{x \in [u,v] : f(x) < g(x)\}$ is open, g equals f on $I \setminus G$ and g is constant on every connected component of G. So, Lemma 5 gives (3). $\square$

Lemma 7. *Let $f \in C(I,I)$. Then in every neighborhood of f there is a piecewise monotone map g with $h(g) \leq h(f)$.*

Proof. Let $\varepsilon > 0$. Take $\delta > 0$ such that $\operatorname{diam} f(J) < \varepsilon$ whenever $\operatorname{diam} J < \delta$. Now take points $\min I = z_1 < z_2 < \cdots < z_k = \max I$ with $|z_i - z_{i+1}| < \delta$ for $i = 1, 2, \dots, k-1$. In each interval $[z_i, z_{i+1}]$ take a point s_i such that $f(s_i) = \max f([z_i, z_{i+1}])$. Let

$$\{z_i : i = 1, 2, \dots, k\} \cup \{s_i : i = 1, 2, \dots, k-1\} = \{x_1, x_2, \dots, x_n\}$$

with $\min I = x_1 < x_2 < \cdots < x_n = \max I$. Since each of the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \dots, n-1$, can be viewed as the interval $[u,v]$ in Lemma 6, we can use the lemma $n-1$ times to get a piecewise monotone map g with $h(g) \leq h(f)$ and $\varrho(f,g) < \varepsilon$. $\square$

Proof of Theorem. Since for $\varphi \in C(I,I)$ we have $h(\varphi) = 0$ if and only if φ is of type at most 2^∞ , it suffices to use Lemma 7 and Lemma 4. $\square$

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