.

DOMINATION OF FUZZY INCIDENCE GRAPHS WITH APPLICATION IN COVID-19 TESTING FACILITY

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Abstract. In this paper, order, size and domination for fuzzy incidence graphs are defined. We explain these concepts with some illustrative examples. We also explore a relationship between strong and weak fuzzy incidence domination for complete fuzzy incidence graphs ($CFIGs$). Furthermore, an application of domination for fuzzy incidence graph (FIG) to properly manage the COVID-19 testing facility is discussed for the illustration of our proposal.

1. **Introduction and Preliminaries**

A graph is an easy way to express information, including links between different entities. The entities are indicated by nodes or vertices and relationships among these nodes are represented by arcs or edges. Zadeh's was the pioneer who gave an idea of fuzzy sets [20]. Let Z be a set. A mapping $\mu : Z \to [0, 1]$ is called a fuzzy subset (FS) of Z. After Zadeh's excellent work fuzzy graphs (FGs) were introduced by Rosenfeld [13]. Before fuzzy sets, the complications in networks were mainly concerned with disconnection rather than the reduction of flow. In speedy networks such as the internet, the issue of reduction of strength is important than the disconnection. Fuzzy graph theory played a significant role in these areas and has plenty of uses in different fields like communication networks, social networks and optimization problems. Yeh and Bang worked separately on FGs [19]. For a comprehensive study on FGs, we may refer to the reader [2, 8]. The fuzzy tree was studied by Sunitha and Vijayakumar [15]. Order and size in FGs were introduced by Gani [4]. Bhutani gave the idea that FGs can be attached to a fuzzy group as an

Key words and phrases. Fuzzy incidence graph; domination number, decision making.

automorphism group [3]. Akram introduced bipolar FGs [1]. In graphs, the notion of domination was first taken place in the game of chess during the 1850s. In Europe, lovers of chess thought about carefully the complication of fixing the fewer numbers of queens that can be laid down on a chessboard so that all the squares are engaged by a queen. Ore and Berge introduced the concept of domination in 1962. Cockayne and Hedetniemi have further studied about domination in graphs [7]. Somasundaram and Somasundaram have initiated domination in FGs by making use of effective edges (EEs) [17]. At the start, they have verified different characteristics of the domination of a simple graph that still holds for FGs. Xavior et al. [18] has talked about domination in FGs but differently. Dharmalingam and Nithya have also expressed domination parameters for FGs [5]. Equitable DN for FGs was introduced by Revathi and Harinarayaman in [12]. The notion of $(1, 2) - domination$ for FGs was given by Sarala and Kavitha in [14]. Gani and Chandrasekaran have talked about domination in FGs by using strong arcs [11]. Strong domination in FGs was introduced by Sunitha and Manjusha [16]. Dinesh gave the notion of $FIGs$ [6]. Mordeson talked about incidence cuts in $FIGs$ [9].

Some of the basic definitions and results are given below for good understanding. These definitions are taken from [6, 10, 17, 20]. A FG with M as the underlying set is a pair $G = (\varphi, \chi)$ where $\varphi : M \to [0, 1]$ is FS, $\chi : M \times M \to [0, 1]$ is a fuzzy relation on the FS φ such that $\chi(u, v) \leq \varphi(u) \wedge \varphi(v)$ for all $u, v \in M$ and M is finite set. $O(G) = \sum_{u \in M} \varphi(u)$ is called order of graph and $S(G) = \sum_{u,v \in M} \chi(u, v)$ is called size of G. A FG is complete if $\chi(u, v) = \varphi(u) \wedge \varphi(v)$ for all $u, v \in V$. A complete FG is represented by K_φ. In a FG, if $\chi(uv) = \varphi(u) \wedge \varphi(v)$ then u dominates v and v dominates u. A subset M of V is named as dominating set (DS) in G if for each v does not belong to M, $\exists\ u \in M$ such that u dominates v. The domination number (DN) of G is the lowest cardinality of a DS among all DSs in G. The DN of G is expressed by $\gamma(G)$ or γ. A DS M of FG is minimal DS if no proper subset of M is a DS of FG. $N(u) = \{v \in V \mid \chi(uv) = \varphi(u) \wedge \varphi(v)\}$ is said to be the neighborhood of u and $N[u] \cup \{u\}$ is called close neighborhood of u.

For a FG we can generalized a degree of a node in distinct methods. The sum of the weights of the EEs incident at node n is said to be the effective degree (ED) of the node n. It is shown by $dE(n)$. $\delta_E(G) = \wedge\{dE(n) \mid n \in V\}$ shows the lowest ED and $\Delta_E(G) = \vee\{dE(n) \mid n \in V\}$ represents the highest ED. The neighborhood degree of n is defined by $\sum_{m \in N(n)} \varphi(m)$and it is represented by $dN(n)$. $\delta_N(G)$ expresses lowest and $\Delta_N(G)$ shows highest neighborhood degree respectively. In a FG a node m is called an isolated node if $\chi(mn) < \varphi(m) \wedge \varphi(n)$ for all $mn \in \chi^*$.

The main motivation of our work is that in FGs, $\chi(uv) = \chi(vu)$ but normally in $FIGs$, $\psi(u, uv) \neq (v, uv)$ this lead us to introduce domination in $FIGs$. For example in FGs, if a vertex u dominates to vertex v then v also dominates u but in $FIGs$ it is not necessary.

The structure of this article is given as: section 1 contains some foundational definitions and expressions of $FIGs$ that are required to know the content. Section 2, carries definitions of order, size and their connection in $FIGs$. In section 3, we talk about fuzzy incidence domination (FID), and complement of $FIGs$. In section 4, we discuss strong FID, weak FID and a relationship among FID, strong and weak FID for $CFIGs$. In section 5, an application of FID is provided.

Let G be a simple graph having node set V and edge set E. Then an incidence graph (IG) is given by $G = (V, E, I)$ where $I \subseteq V \times E$, IG is shown in Figure 1. If (u, uv) is in IG, then (u, uv) is said to be an incidence pair [10].

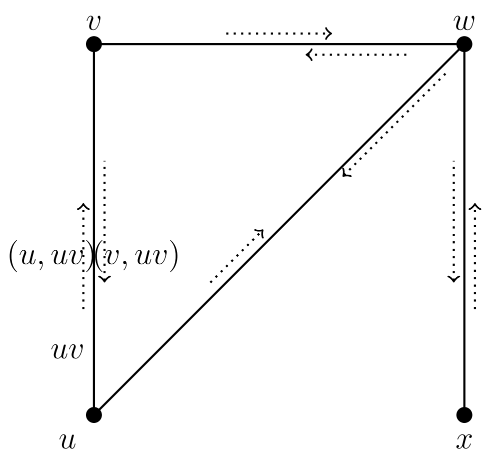


FIGURE 1. A FIG.

Fuzzy incidence (FI) and FIG are defined in [10]. In this paper minimum and maximum operators are represented by $\wedge$ or min and $\vee$ or max, respectively.

Definition 1.1. [10] Consider a graph $G = (V, E)$, φ and χ are FSs of V and E respectively. Assume, $V \times E$ has a FS ψ. If $\psi(v, e) \leq \varphi(v) \wedge \chi(e)$ for every $v \in V$ and $e \in E$, then ψ is named as FI of G and (φ, χ) is known as fuzzy subgraph of G, if ψ is a FI of G, then $G = (\varphi, \chi, \psi)$ is known as a FIG of G.

Remark 1.2. If $\varphi(u) > 0$ then u is in the support of φ where $u \in V$. If $\chi(uv) > 0$ then uv is in the support of χ where $uv \in V \times V$ and if $\psi(u, uv) > 0$ then (u, uv) is in the support of ψ where $(u, uv) \in V \times E$. φ^*, χ^* and ψ^* are representing supports of φ, χ and ψ, respectively [10].

If value of an incidence pair $\psi(u, uv)$ or $\psi(v, vu)$ is not given in the FIG then its value will be equal to zero. Also, two vertices u and v are connected in FIG if there exists a path such that $u, (u, uv), uv, (v, uv), v$ between u and v.

Definition 1.3. [10] A FIG is said to be $CFIG$ if $\psi(i, ij) = \varphi(i) \wedge \chi(ij)$ for each $\psi(i, ij) \in \psi^*$. Also, $\psi(i, ij) = \psi(j, ji)$ for each $i, j \in \varphi^*$. It is denoted by $K^\star$.

Definition 1.4. [9] Let G be a FIG the incidence degree (d_i) of a node $u \in \varphi^*$ is defined as $d_i(u) = \sum_{u \neq v} \psi(u, uv)$.
The lowest d_i of G is defined by $\Omega(G) = min\{d_i(v) | v \in V\}$

The highest d_i of G is defined by $\Delta(G) = max\{d_i | v \in V\}$

2. Relationship between order and size of fuzzy incidence graphs.

In this section, we will discuss the connection between order and size of FIG.

Definition 2.1. Assume $G = (\varphi, \chi, \psi)$ is a FIG. Then $O(G) = \sum_{u \neq v, u, v \in V} \psi(u, uv)$ is called order of G and $S(G) = \sum_{e \in \chi^*} \chi(e)$ is called size of G.

Example 2.2. Assume $G = (\varphi, \chi, \psi)$ is a FIG having $\varphi = \{p, q, r\}$; $\varphi(p) = 0.5, \varphi(q) = 0.6, \varphi(r) = 0.9$; $\chi(pq) = 0.5, \chi(pr) = 0.4, \chi(qr) = 0.5; \psi(p, pq) = 0.4, \psi(q, qp) = 0.3, \psi(p, pr) = 0.3, \psi(r, rp) = 0.4, \psi(q, qr) = 0.5, \psi(r, rq) = 0.4$. Then $O(G) = 2.3$ and $S(G) = 1.4$.

Proposition 2.3. *In a FIG $S(G) \leq O(G)$.*

Proof. Let $G = (\varphi, \chi, \psi)$ be a FIG with one vertex. Then $O(G) = S(G) = 0$. i.e

$$O(G) = S(G). \tag{1}$$

It is a trivial case. Assume G with more than one vertices. $O(G)$ is the sum of all incidence pairs of G. Since incidence pairs are 2 times of edges. Therefore, the total sum of all the membership values of the incidence pairs will always greater than the total sum of all the membership values of the edges.

$$S(G) < O(G). \tag{2}$$

From equations (1) and (2), we get
$S(G) \leq O(G)$. □

Proposition 2.4. *For any FIG the inequality holds: $O(G) \geq S(G) \geq \Delta(G) \geq \Omega(G)$.*

Proof. Assume $G = (\varphi, \chi, \psi)$ is a FIG with non empty vertex set. Since $\Omega(G)$ represents lowest d_i and $\Delta(G)$ denotes highest d_i of G.

$$\Delta(G) \geq \Omega(G). \tag{3}$$

We know $O(G) = \sum_{u \neq v, u, v \in V} \psi(u, uv)$ and $S(G) = \sum_{e \in \chi^*} \chi(e)$.
By definition of size of G, $S(G) = \sum_{e \in \chi^*} \chi(e) \geq \vee\{d_i(v) \mid v \in V\}$
i.e.

$$\Delta(G) \leq S(G). \tag{4}$$

Also, in a FIG, G by proposition 2.3

$$S(G) \leq O(G). \tag{5}$$

From inequalities (3), (4) and (5), we obtained $O(G) \geq S(G) \geq \Delta(G) \geq \Omega(G)$. $\square$

Mordeson has shown $\Sigma_{u\in\sigma^*}(d_i(u)) \leq 2\Sigma_{e\in\mu^*}\mu(e)$ [9]. In his result, there is an inequality. We are going to propose this type of result with equality but in the form of an incidence pairs.

Proposition 2.5. *The d_i sum of all vertices in a FIG is equal to the twice the average sum of all the incidence pairs. i.e.*
$\sum\limits_{v\in\varphi^*} d_i(v) = 2\sum\limits_{u,v\in V}(\frac{\psi(u,uv)+\psi(v,vu)}{2})$.

Proof. Let $G = (\varphi, \chi, \psi)$ be a FIG, where $V = \{v_1, v_2, v_3, ..., v_n\}$, $\varphi \subseteq V$, $\chi \subseteq E$ and $\psi \subseteq V \times E$.

Since $d_i(v) = \sum\limits_{u\neq v}\psi(u, uv)$.
$d_i(v_1) = \psi(v_1, v_1v_2) + \psi(v_1, v_1v_3) + ... + \psi(v_1, v_1v_n)$.
$d_i(v_2) = \psi(v_2, v_2v_1) + \psi(v_2, v_2v_3) + ... + \psi(v_2, v_2v_n)$.
$\vdots$
$d_i(v_n) = \psi(v_n, v_nv_1) + \psi(v_n, v_nv_2) + ... + \psi(v_n, v_nv_{n-1})$.

This implies, $\sum\limits_{v\in V} d_i(v) = d_i(v_1) + d_i(v_2) + ... + d_i(v_n)$.
$\sum\limits_{v\in V} d_i(v) = (\psi(v_1, v_1v_2) + \psi(v_1, v_1v_3) + ... + \psi(v_1, v_1v_n) + \psi(v_2, v_2v_1) + \psi(v_2, v_2v_3) + ... + \psi(v_2, v_2v_n) + ... + \psi(v_n, v_nv_1) + \psi(v_n, v_nv_2) + ... + \psi(v_n, v_nv_{n-1})$.
$\sum\limits_{v\in V} d_i(v) = \frac{2}{2}(\psi(v_1, v_1v_2) + \psi(v_1, v_1v_3) + ... + \psi(v_1, v_1v_n) + \psi(v_2, v_2v_1) + \psi(v_2, v_2v_3) + ... + \psi(v_2, v_2v_n) + ... + \psi(v_n, v_nv_1) + \psi(v_n, v_nv_2) + ... + \psi(v_n, v_nv_{n-1})$.

By rearranging the terms
$\sum\limits_{v\in V} d_i(v) = 2(\frac{\psi(v_1,v_1v_2)+\psi(v_2,v_2v_1)}{2} + \frac{\psi(v_1,v_1v_3)+\psi(v_3,v_3v_1)}{2} + ... + \frac{\psi(v_1,v_1v_n)+\psi(v_n,v_nv_1)}{2} + \frac{\psi(v_2,v_2v_3)+\psi(v_3,v_3v_2)}{2} + \frac{\psi(v_2,v_2v_4)+\psi(v_4,v_4v_2)}{2} + ... + \frac{\psi(v_2,v_2v_n)+\psi(v_n,v_nv_2)}{2}) + ... + \frac{\psi(v_{n-1},v_{n-1}v_n)+\psi(v_n,v_nv_{n-1})}{2})$.
$\sum\limits_{v\in V} d_i(v) = 2\sum\limits_{u,v\in V}(\frac{\psi(u,uv)+\psi(v,vu)}{2})$. $\square$

Example 2.6. Assume $G = (\varphi, \chi, \psi)$ is a FIG given in Figure 2 having $\varphi = \{p, q, r\}$. We have $\sum d_i(v_i) = 2.1$ and $\sum\limits_{u,v \in V} (\frac{\psi(u,uv)+\psi(v,vu)}{2}) = 1.05$. This implies $\sum d_i(v_i) = 2 \sum\limits_{u,v \in V} (\frac{\psi(u,uv)+\psi(v,vu)}{2})$.

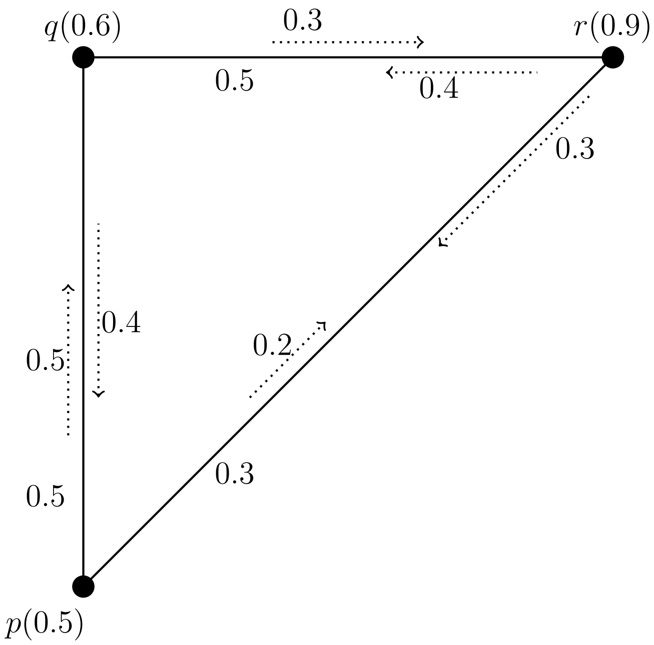


FIGURE 2. A FIG with $\sum d_i(v_i) = 2.1 = 2 \sum\limits_{u,v \in V} (\frac{\psi(u,uv)+\psi(v,vu)}{2}) = 2(1.05)$.

3. **Domination in fuzzy incidence graphs.**

Fuzzy incidence dominating set ($FIDS$) and fuzzy incidence domination number ($FIDN$) for $FIGs$ are discussed in this section.

Definition 3.1. An incidence pair of a FIG is named as an effective incidence pair (EIP) if $\psi(i, ij) = \varphi(i) \wedge \chi(ij)$ for all $i \in V$, $ij \in E$.

Definition 3.2. Open incidence neighborhood (IN) is defined as $IN(i) = \{j \in V \mid \psi(i, ij) = \varphi(i) \wedge \chi(ij)\}$. Closed incidence neighborhood of i is $FIN[i] = FIN(i) \cup \{i\}$.

For a FIG the d_i of a node can be generalized in distinct ways.

Definition 3.3. The effective d_i of a node m is described as $dEIP(m) = \sum \psi(m, mn)$. The minimum effective d_i is denoted by $\delta_{dEIP}(G) = min\{dEIP(m) \mid m \in V\}$. The maximum effective d_i is denoted by $\Delta_{dEIP}(G) = max\{dEIP(m) \mid m \in V\}$.

Definition 3.4. The neighborhood incidence degree (Nd_i) of a node m is expressed as $Nd_i(m) = \sum_{n \in IN(m)} \varphi(n)$. The minimum Nd_i is defined by $\delta_{dIN}(G) = min\{dIN(m) \mid m \in V\}$. The maximum Nd_i is defined by $\Delta_{dIN}(G) = max\{dIN(m) \mid m \in V\}$.

Definition 3.5. A vertex i in a FIG dominates to vertex j if $\psi(i, ij) = \varphi(i) \wedge \chi(ij)$ and a vertex j dominates to i if $\psi(j, ij) = \varphi(j) \wedge \chi(ij)$. The set of these types of vertices is called a $FIDS$ of FIG.

Definition 3.6. The $FIDN$ is the minimum fuzzy incidence cardinality (FIC) of $FIDS$ among all $FIDSs$ in G. It is represented by γ_{FI}.

Example 3.7. Assume $G = (\varphi, \chi, \psi)$ is FIG given in Figure 3 having $FIDSs$ are $Q_1 = \{p, r\}$, $Q_2 = \{p, s\}$ and $Q_3 = \{r\}$ with $FIDN = \gamma_{FI} = \varphi(r) = 0.5$.

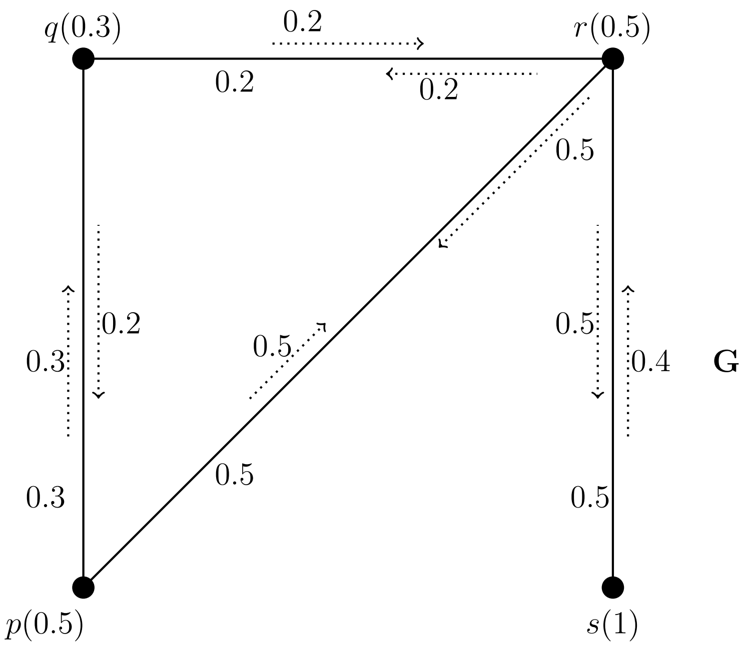


FIGURE 3. FIG with $\gamma_{FI} = 0.5$

Remark 3.8. (1) For any $u, v \in V$, if u dominates v then it is not necessary that v dominates u.

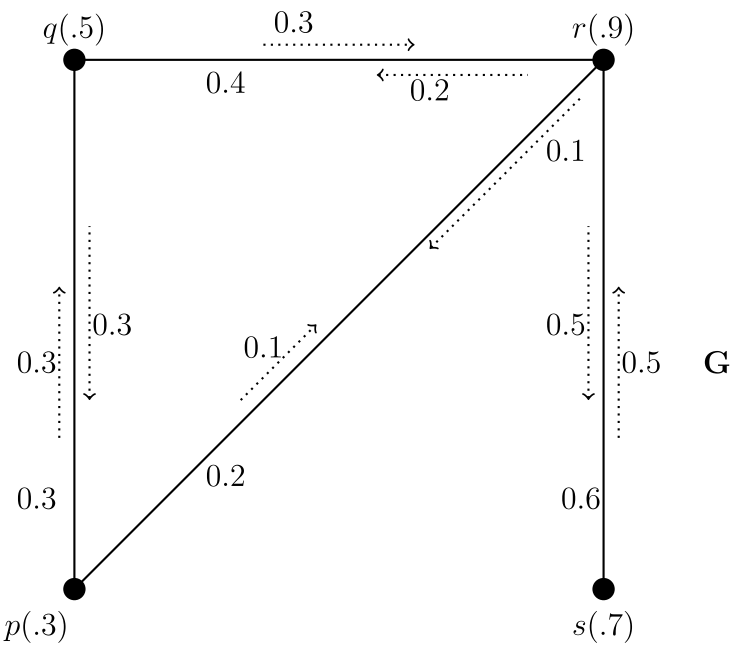
(2) If $\psi(u, uv) < \varphi(u) \wedge \chi(uv)$ $\forall u \in V$, $uv \in E$. This implies V is the unique $FIDS$ of G. Conversely, if V is the only $FIDS$ of G, then $\psi(u, uv) < \varphi(u) \wedge \chi(uv)$ $\forall u \in V$, $uv \in E$.

(3) For $CFIG$, $\{i\}$ is a $FIDS$ for every i belongs to V, we have $\gamma_{FI}(K^{\star}) = min_{x \in V} \varphi(x)$.

Definition 3.9. A node m of FIG is named as an isolated node if $\psi(m, mn) < \varphi(m) \wedge \chi(mn)$ $\forall n \in V - \{m\}$ i.e. $FIN(m) = \emptyset$. Therefore, in FIG no node is dominated by an isolated node but an isolated node dominates to itself.

Definition 3.10. Assume $G = (\varphi, \chi, \psi)$ is a FIG. Then complement of G is indicated by $\overline{G} = \psi'(a, ab) = min(max(\varphi(a), \chi(ab)) - \psi(a, ab), \varphi(a) \wedge \varphi(b) - \chi(ab))$ and the membership values of the vertices in $\overline{G}$ will remain same as in G.

Example 3.11. Assume $G = (\varphi, \chi, \psi)$ is a FIG. Its complement $\overline{G}$ is shown in Figure 4



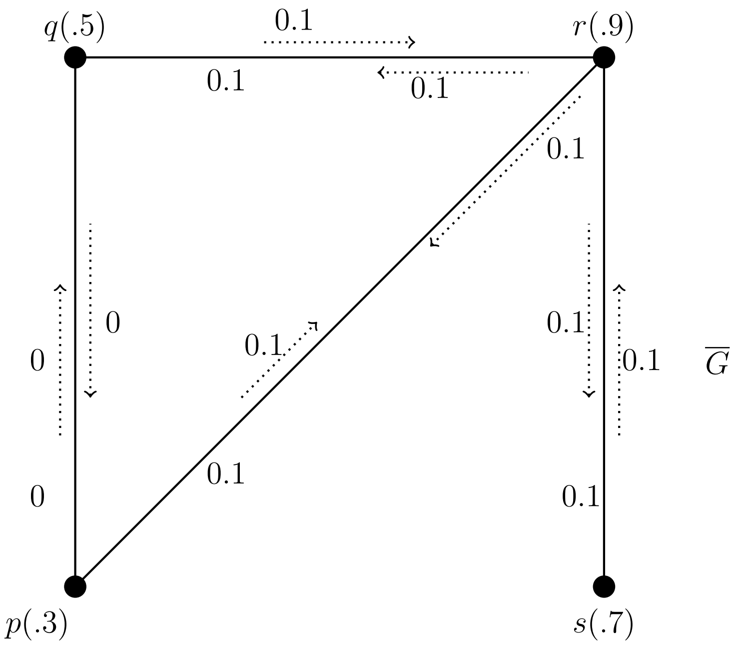


FIGURE 4. Graph **G** is complement to Graph $\overline{G}$

Theorem 3.12. *For any FIG* $2p > \gamma_{FI} + \overline{\gamma}_{FI}$ *where* γ_{FI} *and* $\overline{\gamma}_{FI}$ *are the* $FIDN$ *of* G *and* $\overline{G}$ *respectively.*

Definition 3.13. A $FIDS$ D is called a minimal $FIDS$ of G if no proper subset of D is a $FIDS$ of G.

4. Strong and weak domination in fuzzy incidence graphs

In this section, we have discussed strong and weak FID for $FIGs$ and give different examples to understand these concepts. The results provided in this section are based on [16]. In this view, similar results related to strong $FIDN$ and weak $FIDN$ in $FIGs$ are achieved.

Definition 4.1. Assume G is a FIG and let i and j be the nodes of G. Then i strongly dominates j or j weakly dominates i if the following two conditions are satisfied.

i) $d_i(i) \geq d_i(j)$.

ii) $\psi(i, ij) = \varphi(i) \wedge \chi(ij)$.

We call, j strongly dominates i or i weakly dominates j if $d_i(j) \geq d_i(i)$ and $\psi(j, ji) = \varphi(j) \wedge \chi(ji)$.

Definition 4.2. A set $R \subseteq V$ is a strong $FIDS$ if each node in $V - R$ is strongly fuzzy incidence dominated by at least one node in R. In similar way, R is called a weak $FIDS$ if each node in $V - R$ is weakly fuzzy incidence dominated by at least one node in R.

Definition 4.3. The lowest FIC of a strong $FIDS$ is uttered as the strong $FIDN$ and it is represented by $\gamma_{SFI}(G)$ or γ_{SFI} and the lowest FIC of a weak $FIDS$ is named as the weak $FIDN$ and it is represented by $\gamma_{WFI}(G)$ or γ_{WFI}.

Example 4.4. Assume $G = (\varphi, \chi, \psi)$ is a FIG given in Figure 5 having $\varphi = \{p, q, r, s\}$; $\varphi(p) = 1, \varphi(q) = 0.7, \varphi(r) = 0.5, \varphi(s) = 1$; $\chi(pq) = 0.6, \chi(pr) = 0.4, \chi(qr) = 0.5, \chi(rs) = 0.4$ $\psi(p, pq) = 0.5, \psi(q, qp) = 0.6, \psi(q, qr) = 0.4, \psi(r, rq) = 0.5, \psi(p, pr) = 0.4, \psi(r, rp) = 0.4, \psi(r, rs) = 0.4, \psi(s, sr) = 0.4$ Assume $R = \{r\}$. We have $V - R = \{p, q, s\}$ Here r strongly fuzzy incidence dominates p, q and s because $d_i(r) = 1.3$ is greater than the d_i of all the remaining vertices. i.e. $d_i(p) = 1.0$, $d_i(q) = 1.0$ and $d_i(s) = 0.4$. There is no other strong $FIDS$. Thus the only strong $FIDS$ is $R = \{r\}$. Therefore, $\gamma_{SFI} = .5$. We have weak $FIDS$ is $R_1 = \{p, q, s\}$ with $\gamma_{WFI} = \varphi(p) + \varphi(q) + \varphi(s) = 1 + 0.7 + 1 = 2.7$.

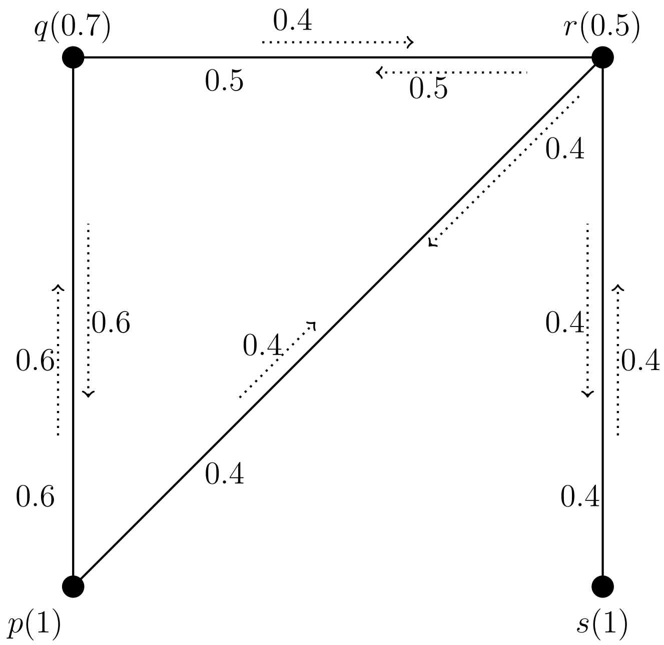


FIGURE 5. G having $\gamma_{SFI} < \gamma_{WFI}$

Remark 4.5. : If G is not a $CFIG$ then $\gamma_{SFI} < \gamma_{WFI}$.

Theorem 4.6. *For any $CFIG$ with $\psi(i,ij) = \varphi(i) \wedge \chi(ij)$ for all $i \in V$, $ij \in E$ the inequality given below is always holds.*

$\gamma_{WFI} \leq \gamma_{SFI}$

Proof. Let $G = (\varphi, \chi, \psi)$ be a $CFIG$ with $\psi(i,ij) = \varphi(i) \wedge \chi(ij)$. Assume for every $w_i \in V$, $\varphi(w_i)$ are same. Since G is $CFIG$ with $\chi(w_iw_j) = \varphi(w_i) \wedge \varphi(w_j)$ for all $w_i, w_j \in V$ and

$\psi(w_i, w_iw_j) = \varphi(w_i) \wedge \chi(w_iw_j)$ for all $w_i \in V$, $w_iw_j \in E$.

Thus, every $w_i \in V$ is strong as well as weak $FIDS$ therefore,

$$\gamma_{SFI} = \gamma_{WFI} \tag{6}$$

Assume for all $w_i \in V$, the $\varphi(w_i)$ are not same. In a $CFIG$ with $d_i(w_i) \geq d_i(w_j)$ from all the nodes one of them strongly dominates all the remaining nodes, if it is smallest among all the nodes then the $FIDS$ with that node is called weak $FIDN$ that is

$\gamma_{WFI} = \varphi(w_i)$ with

$d_i(w_i) \leq d_i(w_j)$ for all w_i, $w_j \in V$ and

$\psi(w_i, w_{ij}) = \varphi(w_i) \wedge \chi(w_i w_j)$ for all $w_i \in V$, $w_i w_j \in E$.

Certainly, the strong $FIDS$ has a node set other than the that node set. This implies

$$\gamma_{WFI} < \gamma_{SFI} \tag{7}$$

from equations (6) and (7), we get

$\gamma_{WFI} \leq \gamma_{SFI}$. □

Example 4.7. Assume $G = (\varphi, \chi, \psi)$ is a $CFIG$ having $\varphi = \{p, q, r\}$; $\varphi(p) = 0.5, \varphi(q) = 0.3, \varphi(r) = 0.8$; $\chi(pq) = 0.3, \chi(pr) = 0.5, \chi(qr) = 0.3$; $\psi(p, pq) = 0.3, \psi(q, qp) = 0.3, \psi(q, qr) = 0.3, \psi(r, rq) = 0.3, \psi(p, pr) = 0.5, \psi(r, rp) = 0.5$. Here $D_1 = \{p\}$ is a strong $FIDS$ which strongly dominates {q,r} and $D_2 = \{r\}$ is another strong $FIDS$ because it also strongly dominates {p,q}. Therefore, $\gamma_{SFI} = 0.5$ and $\gamma_{WFI} = 0.3$.

Theorem 4.8. *For a $CFIG$ the inequalities given below are true*

i) $\gamma_{FI} \leq \gamma_{SFI} \leq$O(G) - highest d_i of G

ii) $\gamma_{FI} \leq \gamma_{WFI} \leq$O(G) - lowest d_i of G

Proof. (i) From definition 4.1, 4.2 and 4.3 we have

$$\gamma_{FI} \leq \gamma_{SFI} \tag{8}$$

We know, $O(G) = p$ the sum of the d_i of FIG

Also,

(9) $O(G)$ - not including the highest d_i of $FIG = O(G) - \Delta(G)$

From equations (8) and (9)

$\gamma_{FI} \leq \gamma_{SFI} \leq$O(G) - highest d_i of G

(ii) From definition 4.1, 4.2 and 4.3 weight of a γ_{FI} of FIG is less than or equal to the γ_{WFI} of FIG, because the vertices of weak $FIDS$ F, it weakly dominates any one of the vertices of $V - F$. Therefore, the weak $FIDN$ will be greater than or equal to the γ_{FI}.

(10) $\gamma_{WFI}(G) \geq \gamma_{FI}(G)$

Also,

(11) $O(G) - \delta(G) = p - \delta(G)$

From equations (10) and (11), we get

$\gamma_{FI} \leq \gamma_{WFI} \leq$O(G) - lowest d_i of G. □

Example 4.9. Assume $G = (\varphi, \chi, \psi)$ is a $CFIG$ having $\varphi = \{p, q, r\}$; $\varphi(p) = 0.8, \varphi(q) = 0.3, \varphi(r) = 0.9$; $\chi(pq) = 0.3, \chi(pr) = 0.8, \chi(qr) = 0.3$; $\psi(p, pq) = 0.3, \psi(q, qp) = 0.3, \psi(q, qr) = 0.3, \psi(r, rq) = 0.3, \psi(p, pr) = 0.8, \psi(r, rp) = 0.8$. $d_i(p) = 1.1, d_i(q) = 0.6, d_i(r) = 1.1, \gamma_{FI} = 0.3, \gamma_{SFI} = 0.8, \gamma_{WFI} = 0.3$, order of $G = 2.8$, highest d_i of $G = 1.1$ and lowest d_i of $G = 0.6$. Hence theorem 4.9 can be verified.

5. **Application of FID for COVID-19 testing facility**

Suppose there are six different medical labs are working in a city for conducting tests of corona virus. Here, in our study we are not mentioning the original names of these labs therefore consider the labs l_1, l_2, l_3, l_4, l_5, and l_6. In $FIGs$, the vertices show the labs and edges show the contract conditions among the labs to share the facilities or test kits. The incidence pairs show the transferring of patients from one lab to another lab due to the lack of resources(machinery, equipment, kits and doctors). $FIDS$ of the graph is the set of labs which perform the tests independently. In this way, we can save the time of patients and to overcome the long traveling of patients by providing the few facilities to the rest of the labs.

Assume $G = (\varphi, \chi, \psi)$ is a FIG shown in Figure 6 having $\varphi = \{l_1, l_2, l_3, l_4, l_5, l_6\}$; $\varphi(l_1) = 0.8, \varphi(l_2) = 0.9, \varphi(l_3) = 0.3, \varphi(l_4) = 0.2, \varphi(l_5) = 0.5, \varphi(l_6) = 0.5$; $\chi(l_1l_2) = 0.6, \chi(l_1l_3) = 0.2, \chi(l_2l_3) = 0.8, \chi(l_3l_4) = 0.2, \chi(l_2l_6) = 0.5, \chi(l_4l_5) = 0.2$; $\psi(l_1, l_1l_2) = 0.5, \psi(l_2, l_2l_1) =$

$0.3, \psi(l_1, l_1l_3) = 0.1, \psi(l_3, l_3l_1) = 0.2, \psi(l_2, l_2l_3) = 0.8, \psi(l_3, l_3l_2) = 0.2, \psi(l_3, l_3l_4) = 0.1, \psi(l_4, l_4l_3) = 0.1, \psi(l_2, l_2l_6) = 0.3, \psi(l_6, l_6l_2) = 0.4, \psi(l_4, l_4l_5) = 0.2, \psi(l_5, l_5l_4) = 0.1$.

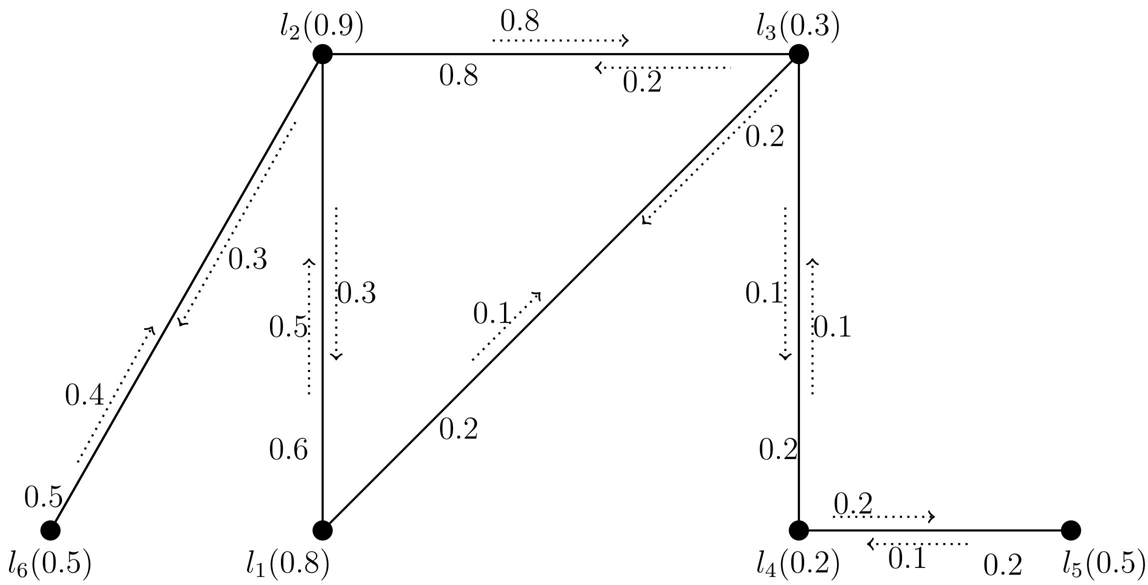


FIGURE 6. A FIG with $\gamma_{FI} = 1.9$.

$FIDS = \{l_2, l_3, l_4, l_6\}$ and $\gamma_{FI} = 1.9$. This shows that patients can visit any one of the lab from this set. Government should provide the resources to the rest of labs only for the proper and easy conduction of tests for corona virus.

6. **Conclusion**

The notion of domination in graphs is vital from theocratical as well as an application's point of view. Different authors have come out with more than thirty-five domination parameters. In this paper, the idea of fuzzy incidence, strong fuzzy incidence, and weak fuzzy incidence domination number is discussed. The results discussed in this paper may be used to study different $FIGs$ invariants. Further work on these ideas will be reported in upcoming papers.

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