Group Theory

Problem Set 1 October 12, 2001

Note: Problems marked with an asterisk are for Rapid Feedback; problems marked with a double asterisk are optional.

1. Show that the wave equation for the propagation of an impulse at the speed of light c,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,,$$

is covariant under the Lorentz transformation

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

2.* The Schrödinger equation for a free particle of mass m is

$$i\hbar\frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \varphi}{\partial x^2}\,.$$

Show that this equation is invariant to the global change of phase of the wavefunction:

$$\varphi \to \varphi' = \mathrm{e}^{i\alpha}\varphi\,,$$

where α is any real number. This is an example of an **internal** symmetry transformation, since it does not involve the space-time coordinates.

According to Noether's theorem, this symmetry implies the existence of a conservation law. Show that the quantity $\int_{-\infty}^{\infty} |\varphi(x,t)|^2\,dx$ is independent of time for solutions of the free-particle Schrödinger equation.

3.* Consider the following sets of elements and composition laws. Determine whether they are groups and, if not, identify which group property is violated.

(a) The rational numbers, excluding zero, under multiplication.

(b) The non-negative integers under addition.

(c) The even integers under addition.

(d) The nth roots of unity, i.e., $\mathrm{e}^{2\pi mi/n}$, for $m = 0, 1, \ldots, n-1$, under multiplication.

(e) The set of integers under ordinary *subtraction*.

4.** The general form of the Liouville equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \big[q(x) + \lambda r(x)\big]y = 0$$

where p, q and r are real-valued functions of x with p and r taking only positive values. The quantity λ is called the eigenvalue and the function y, called the eigenfunction, is assumed to be defined over an interval $[a, b]$. We take the boundary conditions to be

$$y(a) = y(b) = 0$$

but the result derived below is also valid for more general boundary conditions. Notice that the Liouville equations contains the one-dimensional Schrödinger equation as a special case.

Let $u(x;\lambda)$ and $v(x;\lambda)$ be the fundamental solutions of the Liouville equation, i.e. u and v are two linearly-independent solutions in terms of which all other solutions may be expressed (for a given value λ). Then there are constants A and B which allow any solution y to be expressed as a linear combination of this fundamental set:

$$y(x;\lambda) = Au(x;\lambda) + Bv(x;\lambda)$$

These constants are determined by requiring $y(x;\lambda)$ to satisfy the boundary conditions:

$$y(a;\lambda) = Au(a;\lambda) + Bv(a;\lambda) = 0$$

$$y(b;\lambda) = Au(b;\lambda) + Bv(b;\lambda) = 0$$

Use this to show that the solution $y(x;\lambda)$ is unique, i.e., that there is one and only one solution corresponding to an eigenvalue of the Liouville equation.

Group Theory

Problem Set 2 October 16, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Show that, by requiring the existence of an identity in a group G, it is sufficient to require only a *left* identity, $ea = a$, or only a *right* identity $ae = a$, for every element a in G, since these two quantities *must* be equal.

2.* Similarly, show that it is sufficient to require only a *left* inverse, $a^{-1}a = e$, or only a *right* inverse $aa^{-1} = e$, for every element a in G, since these two quantities must also be equal.

3. Show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$.

4.* For the elements $g_1, g_2, \ldots, g_n$ of a group, determine the inverse of the n-fold product $g_1 g_2 \cdots g_n$.

5.* Show that a group is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$. You need to show that this condition is both necessary and sufficient for the group to be Abelian.

6. By explicit construction of multiplication tables, show that there are two distinct structures for groups of order 4. Are either of these groups Abelian?

7.* Consider the group of order 3 discussed in Section 2.4. Suppose we regard the rows of the multiplication table as individual permutations of the elements $\{e, a, b\}$ of this group. We label the permutations π_g by the group element corresponding to that row:

$$\pi_e = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix}, \qquad \pi_a = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}, \qquad \pi_b = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}$$

(a) Show that, under the composition law for permutations discussed in Section 2.3, the multiplication table of the 3-element group is preserved by this association, e.g., $\pi_a \pi_b = \pi_e$.

(b) Show that for every element g in $\{e, a, b\}$,

$$\pi_g = \begin{pmatrix} e & a & b \\ g & ga & gb \end{pmatrix}$$

Hence, show that the π_g have the same multiplication table as the 3-element group.

(c) Determine the relationship between this group and S_3. This is an example of Cayley's theorem.

(d) To which of the operations on an equilateral triangle in Fig. 2.1 do these group elements correspond?

Group Theory

Problem Set 3 October 23, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* List all of the subgroups of any group whose order is a prime number.

2.* Show that a group whose order is a prime number is necessarily cyclic, i.e., all of the elements can be generated from the powers of any non-unit element.

3. Suppose that, for a group G, $|G| = pq$, where p and q are both prime. Show that every proper subgroup of G is cyclic.

4.* Let g be an element of a finite group G. Show that $g^{|G|} = e$.

5. In a quotient group G/H, which set *always* corresponds to the unit "element"?

6. Show that, for an Abelian group, every element is in a class by itself.

7. Show that every subgroup with index 2 is self-conjugate.

Hint: The conjugating element is either in the subgroup or not. Consider the two cases separately.

8.* Consider the following cyclic group of order 4, $G = \{a, a^2, a^3, a^4 = e\}$ (cf. Problem 6, Problem Set 2). Show, by direct multiplication or otherwise, that the subgroup $H = \{e, a^2\}$ is self-conjugate and identify the elements in the factor group G/H.

9.* Suppose that there is an isomorphism ϕ from a group G onto a group G'. Show that the identity e of G is mapped onto the identity e' of G': $e' = \phi(e)$.

Hint: Use the fact that $e = ee$ must be preserved by ϕ and that $\phi(g) = e'\phi(g)$ for all g in G.

Group Theory

Problem Set 4 October 30, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Given a set of matrices $D(g)$ that form a representation a group G, show that the matrices which are obtainable by a similarity transformation $UD(g)U^{-1}$ are also a representation of G.

2.* Show that the trace of three matrices A, B, and C satisfies the following relation:

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$$

3. Generalize the result in Problem 4 to show that the trace of an n-fold product of matrices is invariant under cyclic permutations of the product.

4.* Show that the trace of an arbitrary matrix A is invariant under a similarity transformation UAU^{-1}.

5. Consider the following representation of S_3:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

How can these matrices be permuted to provide an equally faithful representation of S_3? Relate your result to the class identified with each element.

6.* Consider the planar symmetry operations of an equilateral triangle. Using the matrices in Example 3.2 determined from transformations of the coordinates in Fig. 3.1, construct a *three*-dimensional representation of S_3 in the (x, y, z) coordinate system, where the z-axis emanates from the geometric center of the triangle. Is this representation reducible or irreducible? If it is reducible determine the irreducible representations which form the direct sum of this representation.

7. Show that two matrices are simultaneously diagonalizable if and only if they commute.

Group Theory

Problem Set 5 November 6, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1. In proving Theorem 3.2, we established the relation $B_iB_i^\dagger = I$. Using the definitions in that proof, show that this result implies that $B_i^\dagger B_i = I$ as well.

Hint: Show that $B_iB_i^\dagger = I$ implies that $\tilde{A}_i D \tilde{A}_i^\dagger = D$.

2.* Consider the three-element group $G = \{e, a, b\}$ (Sec. 2.4).

(a) Show that this group is Abelian and cyclic (cf. Problem 2, Problem Set 3).

(b) Consider a one-dimensional representation based on choosing $a = z$, where z is a complex number. Show that for this to produce a representation of G, we must require that $z^3 = 1$.

(c) Use the result of (b) to obtain three representations of G. Given what you know about the irreducible representations of Abelian groups (Problem 8, Problem Set 4), are there any other irreducible representations of G?

3.* Generalize the result of Problem 2 to any cylic group of order n.

4.* Use Schur's First Lemma to prove that all the irreducible representations of an Abelian group are one-dimensional.

5.* Consider the following matrices:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

$$c = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Verify that these matrices form a representation of S_3. Use Schur's first Lemma to determine if this representation reducible or irreducible. If reducible, determine the irreducible representations that are obtained from the diagonal form of these matrices.

Group Theory

Problem Set 6 November 13, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Verify the Great Orthogonality Theorem for the following irreducible representation of S_3:

$$e=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad a=\frac{1}{2}\begin{pmatrix}1&-\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix},\quad b=\frac{1}{2}\begin{pmatrix}1&\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$$

$$c=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad d=\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix},\quad f=\frac{1}{2}\begin{pmatrix}-1&-\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$$

2.* Does the following representation of the three-element group $\{e,a,b\}$:

$$e=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad a=\frac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\-\sqrt{3}&-1\end{pmatrix},\quad b=\frac{1}{2}\begin{pmatrix}-1&-\sqrt{3}\\\sqrt{3}&-1\end{pmatrix}$$

satisfy the Great Orthogonality Theorem? Explain your answer.

3.* Specialize the Great Orthogonality Theorem to Abelian groups. When viewed as the components of a vector in a $|G|$-dimensional space, what does the Great Orthogonality Theorem state about the relationship between different irreducible representations? What bound does this place on the number of irreducible representations of an Abelian group?

4.* Consider the irreducible representations of the three-element calculated in Problem 2 of Problem Set 5.

(a) Verify that the Great Orthogonality Theorem, in the reduced form obtained in Problem 3, is satisfied for these representations.

(b) In view of the discussion in Sec. 4.4, would you expect to find any other irreducible representations of this group?

(c) Would you expect your answer in (b) to apply to cyclic groups of any order?

5.* Consider any Abelian group. By using the notion of the order of an element (Sec. 2.4), determine the *magnitude* of every element in a representation. Is this consistent with the Great Orthogonality Theorem?

Group Theory

Problem Set 7 November 20, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Identify the 12 symmetry operations of a regular hexagon.

2. Show that elements in the same class of a group must have the same order.

3.* Identify the 6 classes of this group.

Hint: You do not need to compute the conjugacy classes explicitly. Refer to the discussion for the group S_3 in Example 2.9 and use the fact that elements in the same class have the same order.

4.* How many irreducible representations are there and what are their dimensions?

5.* Construct the character table of this group by following the procedure outlined below:

(a) Enter the characters for the identical and "parity" representations. As in the case of S_3, the characters for the parity representation are either $+1$ or -1, depending on whether or not the the operation preserves the "handedness" of the coordinate system.

(b) Enter the characters for the "coordinate" representation obtained from the action on (x, y) for each group operation. Note that the character is the same for elements in the same class.

(c) Use the products $C_3C_3^2 = E$ and $C_3^3 = E$ to identify the characters for all one-dimensional irreducible representations for the appropriate classes. The meaning of the notation C_n^m for rotations is discussed in Section 5.4.

(d) Use the result of (c) and the products $C_6C_3 = C_2$ to deduce that the characters for the class of C_6 and those for the class of C_2 are the same. Then, use the orthogonality of the *columns* of the character table to compute these characters.

(e) Use the appropriate orthogonality relations for characters to compute the remaining entries of the character table.

Group Theory

Problem Set 8 November 27, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also an orthogonal matrix.

2. Show that the trace of the direct product of two matrices A and B is the product of the traces of A and B:

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$$

3.* Show that the direct product of groups G_a and G_b with elements $G_a = \{e, a_2, \ldots, a_{|G_a|}\}$ and $G_b = \{e, b_2, \ldots, b_{|G_b|}\}$, such that $a_i b_j = b_j a_i$ for all i and j, is a group. What is the order of this group?

4.* Use the Great Orthogonality Theorem to show that the direct product of irreducible representations of two groups is an irreducible representation of the direct product of those groups.

5.* For an n-fold degenerate set of eigenfunctions φ_i, $i, 1, 2, \ldots, n$, we showed show that the matrices $\Gamma(R_\alpha)$ generated by the group of the Hamiltonian,

$$R_\alpha \varphi_i = \sum_{j=1}^{n} \varphi_j \Gamma_{ji}(R_\alpha)$$

form a representation of that group. Show that if the φ_j are chosen to be an orthonormal set of functions, then this representation is *unitary*.

6.* The set of distinct functions obtained from a given function φ_i by operations in the group of the Hamiltonian, $\varphi_j = R_\alpha \varphi_i$, are called **partners**. Use the Great Orthogonality Theorem to show that two functions which belong to different irreducible representations or are different partners in the same unitary representation are orthogonal.

7. Consider a particle of mass m confined to a square in two dimensions whose vertices are located at $(1,1)$, $(1,-1)$, $(-1,-1)$, and $(-1,1)$. The potential is taken to be zero within the square and infinite at the edges of the square. The eigenfunctions φ are of the form

$$\varphi_{p,q}(x,y) \propto \left\{ \begin{array}{c} \cos(k_p x) \\ \sin(k_p x) \end{array} \right\} \left\{ \begin{array}{c} \cos(k_q y) \\ \sin(k_q y) \end{array} \right\}$$

where $k_p = \frac{1}{2}p\pi$, $k_q = \frac{1}{2}q\pi$, and p and q are positive integers. The notation above means that $\cos(k_p x)$ is taken if p is odd, $\sin(k_p x)$ is taken if p is even, and similarly for the other factor. The corresponding eigenvalues are

$$E_{p,q} = \frac{\hbar^2 \pi^2}{8m}(p^2 + q^2)$$

(a) Determine the eight planar symmetry operations of a square. These operations form the group of the Hamiltonian for this problem. Assemble the symmetry operations into equivalence classes.

(b) Determine the number of irreducible representations and their dimensions for this group. Do these dimensions appear to be broadly consistent with the degeneracies of the energy eigenvalues?

(c) Determine the action of each group operation on (x, y).

Hint: This can be done by inspection.

(d) Determine the characters corresponding to the identical, parity, and coordinate representations. Using appropriate orthogonality relations, complete the character table for this group.

(e) For which irreducible representations do the eigenfunctions $\varphi_{1,1}(x, y)$ and $\varphi_{2,2}(x, y)$ form bases?

(f) For which irreducible transformation do the eigenfunctions $\varphi_{1,2}(x, y)$ and $\varphi_{2,1}(x, y)$ form a basis?

(g) What is the degeneracy corresponding to $(p = 6, q = 7)$ and $(p = 2, q = 9)$? Is this a normal or accidental degeneracy?

(h) Are there eigenfunctions which form a basis for each of the irreducible representations of this group?

8.* Consider the regular hexagon in Problem Set 7. Suppose there is a vector perturbation, i.e., a perturbation that transforms as (x, y, z). Determine the selection rule associated with an initial state that transforms according to the "parity" representation.

Hint: The reasoning for determining the irreducible representations associated with (x, y, z) is analogous to that used in Section 6.6.2 for the equilateral triangle.

Group Theory

Problem Set 9 December 4, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Consider the group $\mathrm{O}(n)$, the elements of which preserve the Euclidean length in n dimensions:

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \,.$$

Show that these transformations have $\frac{1}{2}n(n-1)$ free parameters.

2. The condition that the Euclidean length is preserved in two dimensions, $x'^2 + y'^2 = x^2 + y^2$, was shown in lectures to require that

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1 \,.$$

Show that these requirements imply that

$$(a_{11}a_{22} - a_{12}a_{21})^2 = 1 \,.$$

3. Rotations in two dimensions can be parametrized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} .$$

Show that

$$R(\varphi_1 + \varphi_2) = R(\varphi_1)R(\varphi_2)$$

and, hence, deduce that this group is Abelian.

4.* We showed in lectures that a rotation $R(\varphi)$ by an angle φ in two dimensions can be written as

$$R(\varphi) = \mathrm{e}^{\varphi X} \,,$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Verify that

$$\mathrm{e}^{\varphi X} = I\cos\varphi + X\sin\varphi \,,$$

where I is the two-dimensional unit matrix. This shows that $\mathrm{e}^{\varphi X}$ is the rotation matrix in two dimensions.

5.* Consider the two-parameter group

$$x' = ax + b$$

Determine the infinitesimal operators of this group.

6.* Consider the group $C_{\infty v}$ which contains, in addition to all two-dimensional rotations, a reflection plane, denoted by σ_v in, say, the x-z plane. Is this group Abelian? What are the equivalence classes of this group?

Hint: Denoting reflection in the x–z plane by S, show that $SR(\varphi)S^{-1} = R(-\varphi)$.

7. By extending the procedure used in lectures for SO(3), show that the infinitesimal generators of SO(4), the group of proper rotations in four dimensions which leave the quantity $x^2 + y^2 + z^2 + w^2$ invariant, are

$$A_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad A_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad A_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$B_1 = x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}, \qquad B_2 = y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}, \qquad B_3 = z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}$$

8. Show that the commutators of the generators obtained in Problem 7 are

$$[A_i, A_j] = \varepsilon_{ijk}A_k, \qquad [B_i, B_j] = \varepsilon_{ijk}A_k, \qquad [A_i, B_j] = \varepsilon_{ijk}B_k$$

9. Show that by making the linear transformation of the generators in Problem 7 to

$$J_i = \tfrac{1}{2}(A_i + B_i), \qquad K_i = \tfrac{1}{2}(A_i - B_i)$$

the commutators become

$$[J_i, J_j] = \varepsilon_{ijk}J_k, \qquad [K_i, K_j] = \varepsilon_{ijk}K_k, \qquad [J_i, K_j] = 0$$

This shows that locally SO(4)=SO(3)$\otimes$SO(3).

Group Theory

Problem Set 10 December 11, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Prove that a proper orthogonal transformation in an odd-dimensional space always possesses an axis, i.e., a line whose point are left unchanged by the transformation.

2. Prove **Euler's theorem**: The general displacement of a rigid body with one fixed point is a rotation about an axis.

3.* The functions $(x \pm iy)^m$, where m is an integer generate irreducible representations of SO(2). Suppose we now consider the group O(2), where we now allow *improper* rotations. Use Schur's lemma to show that these functions generate irreducible *two*-dimensional representations of O(2) for $m \neq 0$, but a *reducible* representation for $m = 0$.

Hint: The general improper rotation in two dimensions is

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

4. Consider the rotation matrix obtained by rotating an initial set of axes counterclockwise by ϕ about the z-axis, then rotated about the new x-axis counterclockwise by θ, and finally rotated about the new z-axis counterclockwise by ψ. These are the **Euler angles** and the corresponding rotation matrix is

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Verify that the angle of rotation φ of this transformation is given by

$$\cos\left(\tfrac{1}{2}\varphi\right) = \cos\left[\tfrac{1}{2}(\phi + \psi)\right]\cos\left(\tfrac{1}{2}\theta\right)$$

5. Determine the axis of the transformation in Problem 4.

6.* Verify that the direct product of two irreducible representations of SO(3) has the following decomposition

$$\chi^{(\ell_1)}(\varphi)\chi^{(\ell_2)}(\varphi) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi^{(\ell)}(\varphi)$$

This is called the **Clebsch–Gordan series** and provides a group-theoretic statement of the addition of angular momenta.

7.* Determine the corresponding Clebsch-Gordan series for SO(2).

8.* Show that the requirement that $xx^* + yy^*$ is invariant under the complex transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

together with the determinant of this transformation being unity means that the transformation must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $aa^* + bb^* = 1$.